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I, Dewey Estep , hereby submit this original work as part of the requirements for the degree of Doctor of Philosophy in Mathematical Sciences.

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Prime end boundaries of domains in metric spaces and the Dirichlet problem

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Prime end boundaries of domains in metric spaces, and the Dirichlet problem

by

Dewey Estep

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ABSTRACT

Prime end boundaries of domains in metric spaces, and the Dirichlet problem

by

Dewey Estep

Chair: Professor Nageswari Shanmugalingam

Let Ω be a domain in a metric measure space X of bounded geometry. In this thesis we define and investigate the prime end boundary bounded Ω, denoted $\partial_P\Omega$, and attempt to solve the Dirichlet problem on said domains. We show that, in bounded Ω satisfying a certain key assumption, we may solve the Dirichlet problem with prime end boundary data f by using the Perron method and that such a solution coincides with the solution Hf given by the obstacle problem on Ω with obstacle $-\infty$. Here, our key assumption is that every end of Ω has a prime end of Ω which divides it. It is currently unknown if any bounded domains fails to satisfy this assumption. We also create a definition of prime ends for unbounded Ω. By using the sphericalization results of Li and Shanmugalingam in [20], we are able to show that the prime end boundary of an unbounded Ω is homeomorphic to the prime end boundary of the image of Ω under the sphericalization of X. We then show that we may solve the Dirichlet problem for such domains with prime end boundary data f by using the Perron method and that such a solution coincides with the solution Hf given by the appropriate obstacle problem, with the additional assumption that $f - Hf$ extends p-quasicontinuously to 0 on $\partial_P\Omega$.

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CHAPTER 1

Introduction

The classical Dirichlet problem on a domain Ω is to find a function u of appropriate regularity such that $\Delta_p u = 0$ in $\Omega \subset \mathbb{R}^n$ and $u = f$ on $\partial\Omega$ for a specified boundary data $f : \partial\Omega \to \mathbb{R}$, where

$$\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u).$$

The (more interesting) weak version of the Dirichlet problem asks for a function $u \in W^{1,p}(\Omega)$ such that $\Delta_p u = 0$ in a weak sense and $u - f \in W_0^{1,p}(\Omega)$. Here, the space $W^{1,p}(\Omega)$ is the classical Sobolev space for domains in $\mathbb{R}^n$.

It is also possible to pose such a problem for more general metric measure spaces X. To do so, one must introduce an appropriate analogue of the weak derivatives taken in $W^{1,p}$. There are several ways to approach this problem, each attempting to encapsulate a different key property or use of the usual weak derivative. Here we use the concept of an *upper gradient*, first introduced by Heinonen and Koskela in [17]. The upper gradient can be thought of as an analogue of $|\nabla u|$. Functions on X which are in L^p and have L^p upper gradients are called *Newtonian* and are collected in the set $N^{1,p}(X)$, which forms the analogue of $W^{1,p}$ in this setting. The space of Newtonian functions was first studied by Shanmugalingam in [23]. The fact that $N^{1,p}(\mathbb{R}^n)$ and $W^{1,p}(\mathbb{R}^n)$ are equivalent, proved in [23], forms a convincing argument for the appropriateness of this definition.

Of course, even with an adequately defined analogue of a derivative, a poorly behaved metric measure space might refuse to permit interesting results. Therefore, we will restrict ourselves to metric measure spaces of so-called *controlled geometry*. Such spaces are complete, doubling, and support a p-Poincaré inequality. The exact definitions of these properties will be introduced in Chapter 2 of this thesis. Suffice it to say that these properties grant X the structure necessary to allow for meaningful calculus to be performed upon it.

Under these conditions, the Dirichlet problem was formulated and solved in [6]. Additionally, in [7] the Perron method was shown to be appropriate for the construction of solutions for a wide class of boundary data. The Perron method was first introduced in 1923 in [22], and allows for the construction of solutions to a Dirichlet problem with boundary data that is not necessarily continuous. Precise details of the Perron method will be given in Chapter 3.

The above statements and solutions of the Dirichlet problem all considered boundary data prescribed on the metric boundary of the domain Ω under consideration. However, the topological boundary $\partial\Omega$ is sometimes not the correct one to consider. A simple example of this comes in the case of the slit disk, that is, the unit disk in $\mathbb{C}$ with the non-negative reals removed.

Such a domain could be seen to model a circular container filled with viscous fluid into which a thin, non-conductive plate (represented by the slit in the disk) has been inserted. If we allow for the temperatures of the two sides of this plate to be controlled independently, it becomes impossible to model this situation satisfactorily with boundary data given purely on the metric boundary.

Thus, the correct way to model the behavior of the slit is to associate with each part of the slit two points: one to represent the top of the slit and the other the bottom of the slit.

To do this, we turn to prime ends. Prime ends were originally introduced in 1913 by Carathéodory for simply connected planar domains in [9]. When applied to the model of the slit disk, the resulting prime end boundary behaves as desired. A natural question, then, is whether the Dirichlet problem is solvable when posed with prime end boundary data.

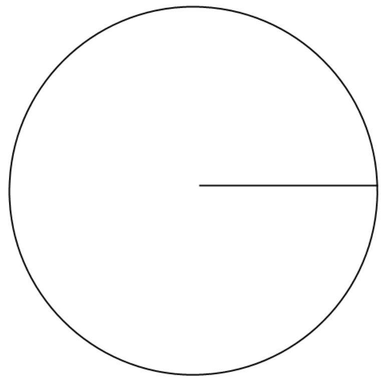

Figure 1.1: The slit disk as viewed with the metric boundary

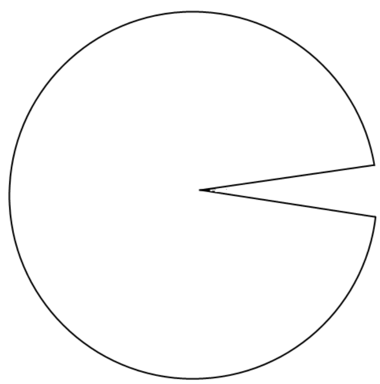

Figure 1.2: The slit disk as viewed with the prime end boundary

Carathéodory's prime ends were defined in terms of curves which 'cut' a space in two, which prevents them from being well-defined in a large class of spaces. This definition has been extended to higher-dimensional Euclidean domains by Näkki in [21] and to arbitrary m-manifolds without boundary by Epstein in [10]. We wish to consider such a Dirichlet problem in more general metric measure spaces, and therefore we must present a definition of prime ends which is valid in these spaces. Such a definition is given in [1]. The definitions therein, while similar to those of

Caratheodory, depend on more general topological concepts and are thus valid in a much wider array of spaces. Specifically, they are valid in metric measure spaces of bounded geometry. As shown below, this new definition does not necessarily agree with Carathéodory's where they are both defined. However, they do agree on the less pathological sections of the boundary.

The Dirichlet problem with prime end boundary data has already been investigated under certain specific conditions. In [2] the problem is solved specifically for the domain in $\mathbb{R}^2$ known as the topologist's comb. Under certain conditions, the prime end boundary is homeomorphic to the Mazurkiewicz boundary, the metric boundary given by the Mazurkiewicz distance. In [5], the Dirichlet problem on bounded domains with Mazurkiewicz boundary data and compact Mazurkiewicz boundary is solved.

However, in more general bounded domains the prime end boundary might be very badly behaved. In fact, the topology associated with the prime end boundary can easily fail to be metrizable, much less Hausdorff. Even when the boundary is metrizable, it may still fail to be compact, and thus remain outside the scope of [5]. It is the Dirichlet problem in these more general bounded domains we wish to discuss in this thesis. We solve the problems in such domains by use of the Perron method.

We will begin by defining the prime end boundary and proving several structural results pertaining to it. Here we will use a slight modification of the definition given in [1]. We do so to allow for more flexibility in the construction of prime ends. Luckily, this modification does not invalidate the important results from [1] that we wish to use.

The key structural result that we wish to prove is the equivalence of compact containment of sets under both the prime end and metric topologies. Such a result would allow us to prove a comparison principle with regards to the prime end boundary, an essential tool in the use of the Perron method. Unfortunately, to prove this key result, we must make an additional assumption about the structure of the prime end boundary of our domain.

At this time, this assumption is not known to hold in general, though it is easy to show that it holds for simply connected bounded planar domains. Since even such domains can still fail to have metrizable prime end boundary, these results are still new regardless of the universality of this assumption.

We then proceed to introduce precisely the terminology and theory behind Newtonian spaces and the Perron method. Using the structural results proved previously, we show that the Perron method produces a solution to the Dirichlet problem in bounded domains with prime end boundary data.

It should be noted that, in general, the functions constructed by using the Perron method are not necessarily unique. Instead, the Perron method yields, in some sense, the smallest and the largest solutions with the prescribed boundary data. Thus, not all boundary data will yield a reasonable solution via the Perron method. Boundary data on the prime end boundary which is *resolutive* (that is, it yields a unique Perron solution) will be considered reasonable boundary data here. We will show that Sobolev and Lipschitz boundary data (and their zero-capacity perturbations) are resolutive.

Finally, we attempt to consider the Dirichlet problem on unbounded domains with prime end boundary data. To do so, we must first define prime ends for such domains, as the definitions presented in [1] and Chapter 2 of this thesis presume the boundedness of our domain. We do so by considering the corresponding domain within the sphericalization of its ambient space. Using this definition, we are able to exploit the structural results of prime ends in bounded domains proved earlier and show that the Perron method also produces a solution to the Dirichlet problem in this case.

CHAPTER 2

The Structure of the Prime Ends of Bounded Domains

In this chapter we will construct the prime end boundary for a bounded domain in a complete doubling metric space. While the notion of prime ends in such a space was introduced in [1], we use a slightly modified definition to allow for more possible prime ends. We will then proceed to prove several structural results regarding the prime end boundary. Of particular interest will be Theorem 2.4.3, as it will be key to our approach of solving the Dirichlet problem in Chapter 3.

2.1 Preliminary notation

For this chapter, we assume that (X, d, μ) is a complete, doubling metric measure space that is quasiconvex. A space X is said to be quasiconvex if there is a constant $C_q \geq 1$ such that whenever $x, y \in X$, there is a rectifiable curve (that is, a curve of finite length) γ with end points x and y such that the length of γ, denoted $\ell(\gamma)$, is at most $C_q\, d(x, y)$. Quasiconvexity is a consequence of the validity of a p-Poincaré inequality on (X, d, μ) when μ is doubling (proved in [3, Theorem 4.32]), both of which properties will be required of our space in later chapters. Thus, the assumption of quasiconvexity here is natural. Furthermore, complete doubling metric spaces are known to be proper. A metric space is proper if closed and bounded subsets of the space are compact. This property is vital, as it will enable us to apply the Arzelá-Ascoli theorem in the proofs of many key results in this chapter.

In addition to the standard metric balls $B(x,r) := \{y \in X \,:\, d(x,y) < r\}$, we will also make use of the *r-neighborhood* of a set for $r > 0$, defined as

$$N(A,r) := \bigcup_{x \in A} B(x,r). \tag{2.1}$$

We will also use the notion of the distance from a point to a set and distance between two sets:

$$\operatorname{dist}(x,A) := \inf\{d(x,y) \,:\, y \in A\}, \quad \operatorname{dist}(A,B) := \inf\{d(x,y) \,:\, x \in A, y \in B\}.$$

Given a connected, open subset Ω of X, we will make use of two additional metrics defined within Ω, the first of which is defined below.

Definition 2.1.1. Let Ω be a bounded connected open subset of X, that is, Ω is a bounded domain. Given $x, y \in \Omega$, the *Mazurkiewicz distance d_M between x and y in Ω* is

$$d_M^\Omega(x,y) = \inf_E \operatorname{diam} E,$$

where the infimum is taken over all connected sets $E \subset \Omega$ with $x, y \in E$.

It is clear that d_M is a metric on Ω. The completion of Ω under d_M is denoted $\overline{\Omega}^M$, with $\partial_M \Omega := \overline{\Omega}^M \setminus \Omega$. The metric d_M^Ω extends naturally to a metric on $\overline{\Omega}^M$; this extended metric will also be denoted by d_M^Ω.

Most often, we will only consider d_M^Ω with respect to a fixed bounded domain Ω, and thus we shall suppress the superscript and simply refer to the metric as d_M.

We now define a similar metric on Ω, measured using the lengths of rectifiable curves. Since X is quasiconvex, any open connected subset of X is rectifiably connected.

With this in mind, we define the *inner distance* below.

Definition 2.1.2. Given a set $\Omega \subset X$, the *inner distance* on Ω is given for $x, y \in \Omega$ by

$$\mathrm{dist}_{inn}^{\Omega}(x, y) = \inf_{\gamma} \ell(\gamma),$$

where the infimum is taken over rectifiable curves γ in Ω with end points x, y.

If Ω is a connected open subset of X, then we know that $\mathrm{dist}_{inn}^{\Omega}$ is a metric on Ω. Given that X is complete and proper, an application of the Arzelà-Ascoli theorem tells us that if $\mathrm{dist}_{inn}^{\Omega}(x, y)$ is finite, then there is a $\mathrm{dist}_{inn}^{\Omega}$-'geodesic' $\gamma_{x,y}^{\Omega}$ connecting x to y in $\overline{\Omega}$ with length $\ell(\gamma_{x,y}^{\Omega}) = \mathrm{dist}_{inn}^{\Omega}(x, y)$. Here, by a $\mathrm{dist}_{inn}^{\Omega}$-geodesic we mean a curve in $\overline{\Omega}$ connecting x to y that appears as a uniform limit of a sequence of length-minimzing curves in Ω connecting x to y. Furthermore, the quasiconvexity of X implies that, if Ω is open, then, for each $x \in \Omega$ with $r = \mathrm{dist}(x, X \setminus \Omega)/C_q$, the two metrics d and $\mathrm{dist}_{inn}^{\Omega}$ are biLipschitz equivalent on $B(x, r)$ with biLipschitz constant C_q.

Unlike with d_M, we will not be suppressing the superscript in $\mathrm{dist}_{inn}^{\Omega}$. In later proofs, it will be important to specify exactly what set we are considering the inner distance within.

Note that (2.1) can be applied to the distances in Definitions 2.1.2 and 2.1.1, with the new r-neighborhoods being denoted $N_{inn}^{\Omega}(x, r)$ and $N_M(x, r)$ respectively.

2.2 Prime Ends

We follow [1] in the construction of prime ends for bounded domains in X, with one key modification. We will address the nature and implications of these changes as the end of the section.

In what follows, $\Omega \subset X$ is a bounded open connected set.

Definition 2.2.1. A set $E \subset \Omega$ is *acceptable* if E is connected and $\overline{E} \cap \partial\Omega$ is nonempty. A sequence $\{E_k\}_{k\in\mathbb{N}}$ of acceptable sets is a *chain* if all of the following conditions hold true:

(a) $E_{k+1} \subset E_k$ for $k \in \mathbb{N}$,

(b) for each $k \in \mathbb{N}$, the distance $\mathrm{dist}_M(\Omega \cap \partial E_k, \Omega \cap \partial E_{k+1}) > 0$,

(c) the *impression* $I(\{E_k\}_k) := \bigcap_{k\in\mathbb{N}} \overline{E_k}$ is a subset of $\partial\Omega$.

Note that $I(\{E_k\}_k)$ is a compact, connected set.

Definition 2.2.2. Given two chains $\{E_k\}_k$ and $\{F_k\}_k$, we say that $\{E_k\}_k$ *divides* $\{F_k\}_k$ if, for each positive integer k, there is a positive integer j_k such that $E_{j_k} \subset F_k$.

The above notion of division gives an equivalence relationship on the collection of all chains; two chains $\{E_k\}_k$ and $\{F_k\}_k$ are equivalent if they both divide each other. Given a chain $\{E_k\}_k$, its equivalence class is denoted $[\{E_k\}_k]$. If two chains $\{E_k\}_k$ and $\{F_k\}_k$ are equivalent, then their impressions are equal. Let this (common) impression be denoted $I[\{E_k\}_k]$. These equivalence classes are called *ends* of Ω. The collection of all ends of Ω is called the *end boundary* $\partial_E\Omega$ of Ω.

Observe also that if a chain $\{E_k\}_k$ divides another chain $\{G_k\}_k$, and $\{F_k\}_k \in [\{E_k\}_k]$, then $\{F_k\}_k$ also divides $\{G_k\}_k$. Furthermore, $\{E_k\}_k$ divides every chain in $[\{G_k\}_k]$. Hence, we may extend the notion of divisibility to ends as well.

Definition 2.2.3. Given two ends $[\{E_k\}_k]$ and $[\{F_k\}_k]$, we say that $[\{E_k\}_k]$ *divides* $[\{F_k\}_k]$ if there exist chains $\{E_k\}_k \in [\{E_k\}_k]$ and $\{F_k\}_k \in [\{F_k\}_k]$ such that $\{E_k\}_k$ divides $\{F_k\}_k$.

We adopt the common algebra notation $[\{E_k\}_k]\Big|[\{G_k\}_k]$ to mean that $[\{E_k\}_k]$ divides $[\{G_k\}_k]$.

Remark 2.2.4. It is shown in [1, Remark 4.5] that additionally requiring acceptable sets to open yields an equivalent definition for ends. That is, every end has a representative chain $\{E_k\}_k$ for which each E_k is open in Ω. We will use this fact as is convenient.

Definition 2.2.5. An end of Ω is a *prime end* if the only end that divides it is itself. The collection of all prime ends of Ω, called the *prime end boundary of* Ω, is denoted $\partial_P\Omega$. The collection of all prime ends of Ω with singleton impression is called the *singleton prime end boundary* and is denoted $\partial_{SP}\Omega$.

We now describe a topology on $\partial_E\Omega \cup \Omega$ that agrees with the topology of Ω. We do so by way of defining a sequential topology on $\partial_E\Omega \cup \Omega$. First, we include the original notions of convergence of sequences in Ω to points in Ω given by the subspace topology inherited from X. Thus the topology of $\partial_E\Omega \cup \Omega$ restricted to Ω will be identical to that of Ω itself. Next, we "stitch" $\partial_E\Omega$ to Ω via the following definitions of convergence.

Definition 2.2.6. Given a sequence $\{x_i\}_i$ in Ω and an end $[\{E_k\}_k] \in \partial_E\Omega$, we say that $x_i \overset{\overline{\Omega}^P}{\to} [\{E_k\}_k]$ if for every positive integer k there is a positive integer i_k such that whenever $i \geq i_k$ we have $x_i \in E_k$.

We next extend the topology to $\partial_E\Omega$ by describing convergence within $\partial_E\Omega$ alone.

Definition 2.2.7. Given a sequence $\{[\{E_k^n\}_k]\}_n$ of ends of Ω and an end $[\{E_k^\infty\}_k]$ of Ω, we say that $[\{E_k^n\}_k] \overset{\overline{\Omega}^P}{\to} [\{E_k^\infty\}_k]$ if for each positive integer k there is a positive integer n_k such that whenever $n \geq n_k$, there is a positive integer j_n with $E_{j_n}^n \subset E_k^\infty$.

Note that a sequence of ends will never converge to a point in Ω.

Definition 2.2.8. Equip the set $\overline{\Omega}^E := \Omega \cup \partial_E\Omega$ with the sequential topology associated with the above notion of limits. Equip the subset $\overline{\Omega}^P := \Omega \cup \partial_P\Omega$ with the subspace topology inherited from $\overline{\Omega}^E$. We call the sets $\overline{\Omega}^E$ and $\overline{\Omega}^P$ the *End Closure of* Ω and the *Prime End Closure of* Ω, respectively.

Though it may be tempting to use the notation $\lim_{n\to\infty} x_n$ to denote a limit end of a sequence, it should be noted that a single sequence (of either points in Ω or $\partial_E\Omega$) may converge to two *different* ends. Thus, such notation would not be well defined. Note that this implies that this topology may not necessarily be Hausdorff, though under certain restrictions it may be metrizable (see Remark 2.2.11 below).

Example 2.2.9. Let $X = \mathbb{R}^2$ and $\Omega = (0,2)^2 \setminus I$, where

$$I = \left(\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 2 - \frac{1}{n}\right] \times \left\{\frac{1}{2n}\right\}\right) \cup \left(\bigcup_{n=1}^{\infty} \left(\left[0, 1 - \frac{1}{n}\right] \cup \left[1 + \frac{1}{n}, 2\right]\right) \times \left\{\frac{1}{2n+1}\right\}\right).$$

Let $E_k = (0, 1 + \frac{1}{k}) \times (0, \frac{1}{2k})$ and $F_k = (1 - \frac{1}{k}, 2) \times (0, \frac{1}{2k+1})$. Then $[\{E_k \cap \Omega\}_k]$ and $[\{F_k \cap \Omega\}_k]$ are distinct prime ends of Ω.

For the sequence $\{(1, \frac{2}{2k+1})\}_k$ of points in Ω, it is clear that $(1, \frac{2}{2k+1}) \xrightarrow{\overline{\Omega}^P} [\{E_k \cap \Omega\}_k]$ and $(1, \frac{2}{2k+1}) \xrightarrow{\overline{\Omega}^P} [\{F_k \cap \Omega\}_k]$.

In fact, this example illustrates that this situation is easy to replicate. Indeed, if there are ever two ends $[\{E_k\}_k]$ and $[\{F_k\}_k]$ such that $E_k \cap F_k$ for every k, we may find a sequence in Ω that converges to both ends.

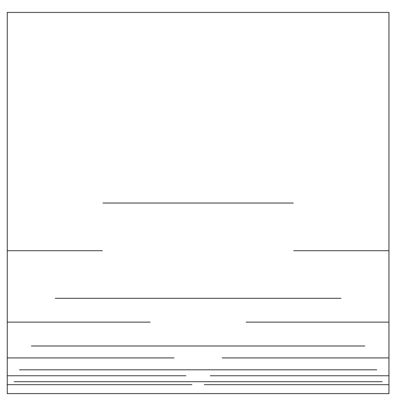

Figure 2.1: The domain described in Example 2.2.9

Remark 2.2.10. The definition of a chain presented in Definition 2.2.1 differs from that presented in [1] only in that condition (b) originally referenced $\operatorname{dist}(\cdot,\cdot)$, instead of $\operatorname{dist}_M(\cdot,\cdot)$. The reason for this change in distance is to allow more flexibility in the definition of chains. In the figure below, the two dashed lines have metric distance zero between them, however they have positive Mazurkiewicz distance between them. One could not create a chain in the sense of [1] with acceptable sets whose boundaries corresponded to the dashed lines. However, there is no real reason why such a chain should be considered undesirable. By using the Mazurkiewicz distance instead, we allow such a chain to be used.

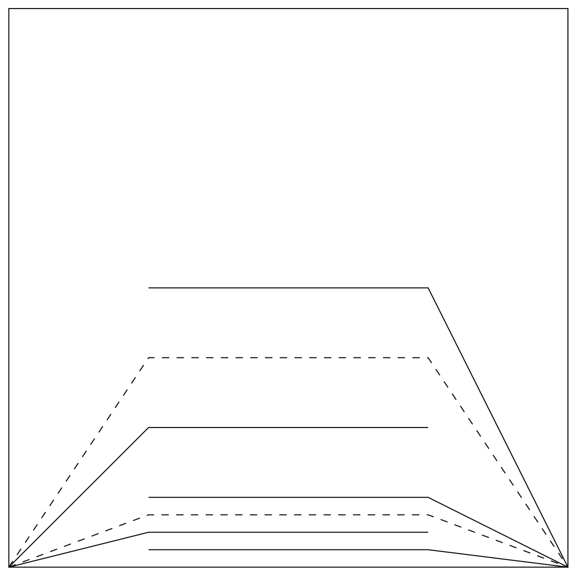

Figure 2.2: An example of a reasonable chain which is invalid in [1]

However, the examples and results of [1] still hold. Indeed, whenever the analog of condition (b) was used in [1] to prove a claim, the key property used was that when $\{E_k\}_{k\in\mathbb{N}}$ is a chain, for each k and points $x \in E_{k+1}$ and $y \in \Omega \setminus E_k$, every connected compact subset of Ω that contains both x and y must have diameter bounded below by a positive number that may depend on k but not on x, y. This is *precisely* the condition given by our version of condition (b), and so the results of [1] hold for our ends as well.

Since $d(\cdot,\cdot) \leq d_M(\cdot,\cdot)$ (where they are mutually defined), we see that any chain in the sense of [1] is a chain in our sense. However, the converse need not be true. Therefore in general we have more chains in the sense of Definition 2.2.1 than does [1].

Thus, we may conceivably have more ends than does [1] and thus an end that might be prime in the setting of [1] (see the definition of prime ends below) may not be prime in our sense. However, given that the notion of Sobolev spaces in the metric setting considered here uses paths extensively (and hence relies on connectivity), the Mazurkiewicz distance seems to be the natural one to consider here.

Sometimes, it may be useful to talk about the closure or boundary of a set $V \subset \overline{\Omega}^P$ with respect to the Prime End topology of Ω. To avoid confusion we will denote the *Prime End closure of V with respect to the Prime End topology on Ω* as $\overline{V}^{P,\Omega}$ and the *Prime End boundary of V with respect to the Prime End topology of $\overline{\Omega}^P$* as $\partial_P^\Omega V$. Note that if $\overline{V} \subset \Omega$, then $\overline{V}^{P,\Omega} = \overline{V}$ and $\partial_P^\Omega V = \partial V$.

Remark 2.2.11. Recall that by $\partial_{SP}\Omega$ we mean the collection of all prime ends of Ω whose impressions contain only one point. Recall also the Mazurkiewicz boundary $\partial_M \Omega$ of Ω from Definition 2.1.1. Though $\overline{\Omega}^P$ admits no metric, it is shown in [1, Theorem 9.5] that there is a homeomorphism $\Phi : \Omega \cup \partial_{SP}\Omega \to \overline{\Omega}^M$ such that $\Phi|_\Omega$ is the identity map and $\Phi|_{\partial_{SP}\Omega} : \partial_{SP}\Omega \to \partial_M\Omega$. It follows that $\Omega \cup \partial_{SP}\Omega$ is metrizable via the pullback of the metric d_M. So, for $x, y \in \Omega \cup \partial_{SP}\Omega$, by $d_M(x, y)$ we truly mean $d_M(\Phi(x), \Phi(y))$.

Remark 2.2.12. Given a set $G \subset \Omega$, we define

$$G^P := G \cup \{[\{E_k\}_k] \in \partial_P\Omega \mid \text{for some } j,\ E_j \subset G\}.$$

It was shown in [1, Proposition 8.5] that the collection of sets

$$\{G, G^P \mid G \subset \Omega \text{ is open}\}$$

forms a basis for the topology on $\overline{\Omega}^P$. Note that given the above definition of G^P, we have $\overline{\Omega}^P = \Omega^P$. In the next few sections, we will focus on the sequential definition of this topology. In later sections, the above natural basis will prove invaluable in making our results more intuitive.

It should be noted that, though it is tempting to assume so, not every open set in the topology of $\overline{\Omega}^P$ is of the from $G_1 \cup G_2^P$ for some open sets $G_1, G_2 \subset \Omega$. As shown in the below examples, there are open sets which do not look precisely like those in the basis.

Example 2.2.13. Let $\hat{\Omega}$ be the so-called double harmonic comb, defined as $((0,1) \times (0,1)) \setminus I$, with

$$I = \left(\bigcup_{n=1}^{\infty} \left[0, \frac{3}{4} \right] \times \left\{ \frac{1}{2n} \right\} \right) \cup \left(\bigcup_{n=1}^{\infty} \left[\frac{1}{4}, 1 \right] \times \left\{ \frac{1}{2n+1} \right\} \right).$$

Consider $\Omega := \hat{\Omega} \times (0,1) \subset \mathbb{R}^3$. For each $a, b \in [0,1]$, let $I_{a,b}$ be the closed line segment connecting $(\frac{1}{4}, 0, a)$ to $(\frac{3}{4}, 0, b)$. Then, if we let

$$E_k^{a,b} = \left\{ x \in \Omega \mid \mathrm{dist}(x, I^{a,b}) < \frac{1}{k} \right\},$$

then $[\{E_k^{a,b}\}_k]$ is a prime end with impression $I_{a,b}$. Consider

$$G_1 = \hat{\Omega} \times \left(0, \frac{2}{3} \right) \text{ and } G_2 = \hat{\Omega} \times \left(\frac{1}{3}, 1 \right).$$

If, for some open sets $G_3,\ G_4 \subset \Omega$, we have that

$$G_1^P \cup G_2^P \subset G_3^P \cup G_4,$$

then it must be the case that $[\{E_k^{0,1}\}_k] \in G_3^P$. However, $[\{E_k^{0,1}\}_k] \notin G_1^P \cup G_2^P$. Thus $G_1^P \cup G_2^P \neq G_3^P \cup G_4$ for any open $G_3,\ G_4 \subset \Omega$.

Definition 2.2.14. We say that a point $x_0 \in \partial\Omega$ is *accessible from* Ω if there is a curve $\gamma : [0,1] \to \overline{\Omega}$ such that $\gamma(1) = x_0$ and $\gamma([0,1)) \subset \Omega$. We say that a point $x_0 \in \partial\Omega$ is *accessible through the chain* $\{E_k\}_k$ if this γ additionally has the property that, for each positive integer k there is some $0 < t_k < 1$ with $\gamma([t_k, 1)) \subset E_k$. The curve γ is said to *access* x_0 *through* $\{E_k\}$.

It is easy to see that if x_0 is accessible through $\{E_k\}_k$ and $\{F_k\}_k \in [\{E_k\}_k]$, then x_0 is accessible through $\{F_k\}_k$ as well. Furthermore, $x_0 \in I[\{E_k\}_k]$. Thus, we can extend the above definitions to ends.

It was shown in [1] that if $z_0 \in \partial\Omega$ is accessible, then it is accessible through some prime end $[\{E_k\}_k]$ with $I[\{E_k\}_k] = \{x_0\}$. In addition, for all singleton prime ends $[\{E_k\}_k] \in \partial_{SP}\Omega$, the point in $I[\{E_k\}_k]$ is accessible through $[\{E_k\}_k]$. The equivalence of singleton prime ends and accessible points of $\partial\Omega$ is an important fact we will make use of here.

Additionally, as ends with singleton impression are quite well-behaved, it is useful to have a simple canonical form of representative chains of such ends. We present such a form as a lemma here.

Lemma 2.2.15. *Let $\{E_k\}_k$ be a chain such that $I(\{E_k\}_k) = \{x_0\}$, that is, the chain has singleton impression. Then $[\{E_k\}_k]$ is a prime end, and may be represented by a chain $\{F_k\}_k$, where each F_k is a connected component of $B(x_0, \frac{1}{k}) \cap \Omega$.*

Proof. We will prove the second part of the lemma, for then the first part follows from [1, Lemma 7.3] and [1, Corollary 7.11] (see also the discussion in [1, Section 10] and [7]).

Let γ be the curve in Ω which accesses x_0 through $\{E_k\}_k$. Let F_k be the connected component of $B(x_0, \frac{1}{k}) \cap \Omega$ which contains the tail end of γ. We now prove that $\{F_k\}$ divides $\{E_k\}_k$. Since $[\{E_k\}_k]$ is prime, this will imply that $[\{F_k\}_k] = [\{E_k\}_k]$.

Let $E_j \in \{E_k\}_k$. Assume that there is no ℓ such that $F_\ell \subset E_j$. Let x_ℓ be an element of $F_\ell \setminus E_j$. Since F_ℓ contains the tail end of γ and $F_\ell \setminus E_j$ is nonempty, it must be the cast that $\partial E_j \cap F_\ell$ is also nonempty. In fact, $\partial E_i \cap F_\ell$ is nonempty for all $i > j$. Thus,

$$\mathrm{dist}_M(\partial E_j \cap \Omega, \partial E_{j+1} \cap \Omega) \leq \mathrm{dist}_M(x_\ell, \partial E_j \cap \Omega) + \mathrm{dist}_M(x_\ell, \partial E_{j+1} \cap \Omega) \leq \frac{2}{\ell}.$$

Since this holds for all ℓ, this contradicts the fact that $\{E_k\}_k$ is a chain. Thus, $\{F_k\}_k$ divides $\{E_k\}_k$. $\square$

Remark 2.2.16. The above lemma immediately tells us that the diameter of sets in a chain representing an $[\{E_k\}_k] \in \partial_{SP}$ tends to zero. In fact, by choosing an appropriate subsequence of sets in such a representative chain, we can control exactly how quickly these diameters shrink. This will be useful in making certain results more readable later.

However, as the example below shows, for some domains Ω not all points in $\partial\Omega$ are accessible from Ω, and it is not true that $\partial_P\Omega$ is always compact. This has implications to the application of the Perron method in solving Dirichlet problems for the boundary $\partial_P\Omega$, and the goal of this approach is to find a way to overcome this lack of compactness; the key lemma in this direction is Lemma 2.3.7.

Example 2.2.17. Let Ω be the double harmonic comb used in the construction of Example 2.2.13. As can be seen, no points in $[0,1]\times\{0\}$ are accessible from within Ω, and in fact no prime ends have impressions which contain points in $\left([0,\frac{1}{4}) \cup (\frac{3}{4},1]\right) \times \{0\}$. Thus, any sequence of prime ends corresponding to the points $\{\frac{1}{8}\} \times \{\frac{1}{2n}\}$ has no convergent subsequence, and thus $\partial_P\Omega$ is not compact.

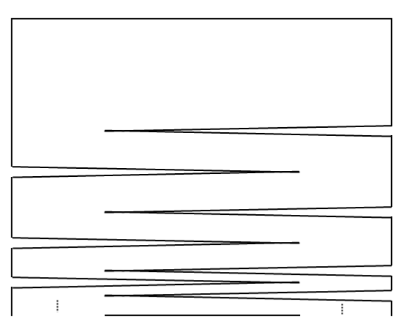

Figure 2.3: The double comb as viewed by the prime end topology

Definition 2.2.18. Let $V \subset \Omega$ be an open connected set. We say that a point $x_0 \in \partial\Omega$ is *accessible from the side of V* if there is a curve $\gamma : [0,1] \to \overline{\Omega}$ such that

$\gamma([0,1)) \subset \Omega$, $\gamma(1) = x_0$, and for each positive integer n there is a real number t_n with $1 - \frac{1}{n} < t_n < 1$ such that $\gamma(t_n) \in V$. We say that a chain $\{E_k\}_k$ of Ω is *from the side of* V if $E_k \cap V$ is non-empty for each positive integer k.

Note that if $\{E_k\}_k$ is from the side of V, and $\{F_k\}_k \in [\{E_k\}_k]$, then $\{F_k\}_k$ is also from the side of V. Hence the property of being *from the side of* V is inherited from chains by ends. Additionally, we can easily see that if a point $x_0 \in \partial\Omega$ is accessible through an end $[\{E_k\}_k]$ which is from the side of V, then x_0 is immediately accessible from the side of V.

Remark 2.2.19. When we discuss curves γ that are locally rectifiable, we assume that γ is essentially arc-length parametrized; that is, $\gamma : [0, \infty) \to X$ such that $\gamma|_{[0,\ell(\gamma))}$ is arc-length parametrized, and if $\ell(\gamma) < \infty$, then for $t \geq \ell(\gamma)$ we have $\gamma(t) = \gamma(\ell(\gamma))$. We call such parametrizations *standard parametrizations.*

Note that in Definitions 2.2.14 and 2.2.18, we could take γ to be maps from $[0, \infty)$ rather than from $[0, 1]$. In this case, in Definition 2.2.18 we require $\ell(\gamma) - 1/n < t_n$ whenever $\ell(\gamma) < \infty$, and $n < t_n < \ell(\gamma)$ when $\ell(\gamma) = \infty$, rather than $1 - 1/n < t_n < 1$.

2.3 The structure of the end and prime end boundaries

In this section we discuss some structural results related to the prime end boundary. We first state two elementary lemmas regarding the geometry of chains. The proofs of these lemmas use the properness of X (that is, closed and bounded subsets of X are compact).

Lemma 2.3.1. *Given a chain $\{E_k\}_k$, for every $\varepsilon > 0$ there is an acceptable set $E_j \in \{E_k\}_k$ such that*

$$E_j \subset N\Big(I(\{E_k\}_k), \varepsilon\Big).$$

Proof. Assume this does not hold. Then, for every $j \in \mathbb{N}$, the set $F_j := \overline{E_j} \setminus N\Big(I(\{E_k\}_k, \varepsilon)\Big)$ is nonempty. Then $\{F_j\}$ is a decreasing sequence of compact sets with

$\overline{F_j} \cap N\Big(I(\{E_k\}), \varepsilon/2\Big) = \emptyset$. Then $F := \bigcap_{j\in\mathbb{N}} \overline{F_j}$ is nonempty and $F \cap N\Big(I(\{E_k\}), \varepsilon/2\Big) = \emptyset$ as well. But $F \subset \bigcap_{j\in\mathbb{N}} \overline{E_j} = I(\{E_k\}_k)$, which is a contradiction. $\square$

Lemma 2.3.2. *Let $\{E_k\}_k$ be a chain. Then, for every $\varepsilon > 0$ and integer k, there is a connected component C_k^ε of $N(I(\{E_k\}_k), \varepsilon) \cap E_k$ such that $I(\{E_k\}_k) \subset \overline{C_k^\varepsilon}$.*

Proof. Assume otherwise. Then there is an integer k and $\varepsilon > 0$ such that the set $N(I(\{E_k\}_k, \varepsilon) \cap E_k$ has no connected components containing $I(\{E_k\}_k)$ in their closure. Then, by Lemma 2.3.1, there is a $j > k$ such that $E_j \subset N(I(\{E_k\}_k), \varepsilon)$. Since $E_j \subset E_k$, $E_j \subset N(I(\{E_k\}_k), \varepsilon) \cap E_k$. Since E_j is connected, it lives within a connected component of $N(I(\{E_k\}_k), \varepsilon) \cap E_k$. Therefore, $I(\{E_k\}) \not\subset \overline{E_j}$, which is a contradiction.

$\square$

Next we prove two useful lemmas about the topology on $\overline{\Omega}^P$.

Lemma 2.3.3. *If $\{x_k\}_k$ is a sequence of points in Ω and $[\{E_k\}_k] \in \partial_E\Omega$ such that $x_k \xrightarrow{\overline{\Omega}^E} [\{E_k\}_k]$, then no subsequence of $\{x_k\}_k$ has a limit point in Ω.*

Proof. Note that $\bigcap_k \overline{E_k} \subset \partial\Omega$ and, for each positive integer j, the tail-end of the sequence $\{x_k\}_k$ lies in E_j. Therefore, every cluster point of $\{x_k\}_k$ must lie in $\bigcap_k \overline{E_k} \subset \partial\Omega$. $\square$

Lemma 2.3.4. *If $U \subset \overline{\Omega}^P$ is an open set in the prime end topology such that $\partial_P\Omega \subset U$, then for each $[\{E_k\}_k] \in \partial_P\Omega$ and for each $\{E_k\}_k \in [\{E_k\}_k]$, there is a positive integer k_U such that $E_{k_U} \subset U$.*

Proof. We prove this lemma by contradiction. Suppose that $\{E_k\} \in [\{E_k\}_k] \in \partial_P\Omega$ such that for each positive integer k we have $E_k \not\subset U$, that is, we can find $x_k \in E_k \setminus U$. It then follows that $\{x_k\}_k$ is a sequence in Ω with $x_k \to [\{E_k\}_k]$. However, since U is open in the sequential topology of $\overline{\Omega}^P$ and $[\{E_k\}_k] \in U$, we must necessarily have a positive integer k_U such that whenever $k \geq k_U$, $x_k \in U$, which contradicts the choice of $x_k \in E_k \setminus U$. $\square$

Next, we prove a useful relation between $\partial_{SP}\Omega$ and $\partial_P\Omega$.

Theorem 2.3.5. *With respect to the prime end topology on $\overline{\Omega}^P$, $\partial_{SP}\Omega$ is dense in $\partial_P\Omega$.*

Remark 2.3.6. In the following proof, we begin by assuming that X is a geodesic space. This may seem, at first glance, to be an extremely restrictive assumption. However, note that all properties of ends under consideration in this proof, namely the rectifiability of curves and positivity of distance, are properties which are preserved under a biLipschitz change in the metric. Since we have already assumed that X is quasiconvex, such a biLipschitz change may be made to transform X into a geodesic space. This is an assumption we will make more than once in this work, and so we collect our justification here for easy reference.

Proof of Theorem 2.3.5. As justified above, we may assume that X is a geodesic space.

Given a prime end $[\{E_k\}_k] \in \partial_P\Omega \setminus \partial_{SP}\Omega$, fix a representative chain $\{E_k\}_k$ of $[\{E_k\}_k]$ such that $E_n \subset N(I[\{E_k\}_k], \frac{1}{n})$. Choose a sequence $\{x_n\}_n$ in Ω such that $x_n \in E_n$ for each positive integer n.

For each x_n, let $R_n = \operatorname{dist}(x_n, X \setminus \Omega)$ and pick $y_n \in \overline{B(x_n, R_n)} \cap \partial\Omega$. Note that, since $x_n \in N(I[\{E_k\}_k], \frac{1}{n})$, we have that $R_n \leq \frac{1}{n}$.

Since X is a geodesic space and $B(x_n, R_n) \subset \Omega$, there is a geodesic $\gamma_n : [0, R_n] \to \overline{\Omega}$ from x_n to y_n such that $\gamma_n([0, R_n)) \subset B(x_n, R_n) \subset \Omega$. Therefore, y_n is accessible and there is a prime end $[\{F_k^n\}_k] \in \partial_{SP}\Omega$ such that $I[\{F_k^n\}_k] = \{y_n\}$ and γ_n accesses y_n through $[\{F_k^n\}_k]$ (see Definition 2.2.14). Though not relevant at the moment, for future use in the proof of Proposition 3.3.15, we note that, since $B(x_n, R_n)$ is connected, $d_M(x_n, [\{F_k^n\}_k]) = R_n \leq \frac{1}{n}$. Furthermore, as mentioned in Remark 2.2.16 we can choose F_k^n so that $\operatorname{diam}(F_k^n) \leq 1/k$.

We now prove that $[\{F_k^n\}_k] \xrightarrow{\overline{\Omega}^P} [\{E_k\}_k]$. Suppose this is not the case. Then there is a positive integer K such that, for each positive integer n, there is an integer $j_n \geq n$ so that for each positive integer k we can find a point $z_{j_n} \in F_k^{j_n} \setminus E_K$. The choice of z_{j_n} does indeed depend on k as well, but since we next fix a choice of positive integer k, we do not indicate the dependance of z_{j_n} on k in the notation. Indeed, we now choose $k \geq 2K + 2n$.

For $n \geq 2K$, we have $x_{j_n} \in E_{K+1} \cap \gamma_{j_n}$, and a set $\beta_{j_n} = \gamma_{j_n} \cup F_k^{j_n}$, containing x_{j_n} and z_{j_n}, with diameter $\operatorname{diam}(\beta_{j_n}) \leq 1/n + 1/k \leq 2/n$. We now show that β_{j_n} is connected. We do not claim here that $x_{j_n} \in F_k^{j_n}$, but note that a point in γ_{j_n} lies in $F_k^{j_n}$, and a compact subcurve of γ_{j_n} therefore connects x_{j_n} to this point. Hence β_{j_n} is connected. Thus we have a point $z_{j_n} \in \beta_{j_n}$ that lies outside E_K, and a point $x_{j_n} \in E_{K+1} \cap \beta_{j_n}$. It follows that $\operatorname{dist}_M(\Omega \cap \partial E_K, \Omega \cap \partial E_{K+1}) \leq 2/n$ for sufficiently large n. Letting $n \to \infty$ we obtain that $\operatorname{dist}_M(\Omega \cap \partial E_K, \Omega \cap \partial E_{K+1}) = 0$, which violates the definition of a chain. Hence we know that $[\{F_k^n\}_k] \overset{\overline{\Omega}^P}{\to} [\{E_k\}_k]$, completing the proof of the theorem. $\square$

The next lemma provides a connection between locally rectifiable curves of infinite length and ends that are, in some sense, from the side of those curves.

Lemma 2.3.7. *Let Ω be a bounded domain in X. Suppose that γ is a curve in Ω such that*

$$I(\gamma) := \bigcap_{n \in \mathbb{N}} \overline{\gamma((n, \infty))} \subset \partial\Omega,$$

and set

$$E(\gamma) := \{[\{F_k\}_k] \in \partial_E \Omega \ |\forall k \in \mathbb{N},\ \exists t_k \text{ such that } \gamma([t_k, \infty)) \subset F_k\}.$$

As in Assumption 2.3.8 below we consider the order relation $\leq$ on $E(\gamma)$ defined by $x \leq y$ if and only if $x|y$. Then $E(\gamma)$ has a minimal (or, least) element $[\{E_k\}_k]$. Furthermore, for each $[\{F_k\}_k] \in E(\gamma)$ we have $[\{E_k\}_k]$ divides $[\{F_k\}_k]$ and $I[\{E_k\}_k] =$

$I(\gamma)$.

Proof. For each positive integer k let E_k denote the connected component of the set $N_M(\gamma((k,\infty)), 1/k) \cap \Omega$ that contains the tail-end $\gamma((k,\infty))$ of γ. Then $\{E_k\}_k$ is clearly a chain, as

$$\mathrm{dist}_M(\partial E_k \cap \Omega, \partial E_{k+1} \cap \Omega) \geq \frac{1}{k} - \frac{1}{k+1} > 0.$$

We will show that the end corresponding to the chain $\{E_k\}_k$ should be a minimal end in $E(\gamma)$.

It is easily seen that $[\{E_k\}_k] \in E(\gamma)$. It suffices to show that whenever $[\{F_k\}_k] \in E(\gamma)$, the end $[\{E_k\}_k]$ divides $[\{F_k\}_k]$. To do so, let $[\{F_k\}_k] \in E(\gamma)$. We want to show that given a positive integer k there is a positive integer j_k such that $E_{j_k} \subset F_k$.

Suppose that the above is not true. Then for each positive integer j the set $E_j \setminus F_k$ is non-empty. By the construction of E_j, for any $x \in E_j \setminus F_k$ there is a real number $t_x \in (j,\infty)$ with $d_M(x, \gamma(t_x)) < 1/j$. Note that by the definition of chains, $\mathrm{dist}_M(\partial F_k \cap \Omega, \partial F_{k+1} \cap \Omega) > 0$. So we can choose a positive integer J such that $1/J < \mathrm{dist}_M(\partial F_k \cap \Omega, \partial F_{k+1} \cap \Omega)$. Consider $j \geq J$, and fix $x_j \in E_j \setminus F_k$, and set $t_j := t_{x_j}$. We then have

$$d_M(x_j, \gamma(t_j)) < 1/j \leq 1/J < \mathrm{dist}_M(\partial F_k \cap \Omega, \partial F_{k+1} \cap \Omega),$$

It follows now from the fact that $x_j \notin F_k$ that $\gamma(t_j) \notin F_{k+1}$. Consequently, for each positive integer $j > J$ we can find a real number $t_j \geq j$ such that $\gamma(t_j) \notin F_{k+1}$. Thus the tail end of γ can lie in F_{k+1}, which violates the fact that $[\{F_k\}_k] \in E(\gamma)$.

Hence we can conclude that necessarily there is some positive integer j_k such that $E_{j_k} \subset F_k$, that is, the end $[\{E_k\}_k]$ divides $[\{F_k\}_k]$, concluding the proof. $\square$

In addition to our previous assumptions on X, we also will assume for the remainder of the work that the domain Ω fulfills the following property:

Assumption 2.3.8. *For every collection $\mathcal{F}$ of ends that is totally ordered by division such that $x \le y$ if and only if $x|y$, there is an end $[\{G_k\}_k]$ such that $[\{G_k\}_k] \le [\{F_k\}_k]$ for every $[\{F_k\}_k] \in \mathcal{F}$.*

The above assumption essentially states that we assume that the collection of all ends of Ω satisfies the hypotheses of Zorn's lemma.

Should Ω be a simply connected bounded planar domain, the above condition is seen to hold true. The proof of this fact there goes through the Riemann mapping theorem; in more general settings it is not clear whether the above condition automatically holds. However, in many situations this condition is directly verifiable. If $\partial_{SP}\Omega$ is compact, then by Theorem 2.3.5 we know that $\partial_P\Omega = \partial_{SP}\Omega$, and in this case the fact that above assumption holds is a consequence of the results found in [1, Section 7]. Indeed, by the results in [1], it follows that given an end $[\{E_k\}_k]$, every point in $I[\{E_k\}_k]$ is accessible through $[\{E_k\}_k]$ by rectifiable curves, and hence a prime end from $\partial_{SP}\Omega$ divides $[\{E_k\}_k]$.

Under the Assumption 2.3.8, we have the following fact about $\overline{\Omega}^E$.

Theorem 2.3.9. *Suppose that Ω is a bounded domain satisfying Assumption 2.3.8. Let $[\{E_k\}_k]$ be an end of Ω. Then there is a prime end $[\{F_k\}_k]$ of Ω that divides $[\{E_k\}_k]$.*

Proof. Consider the set $\mathcal{E}$ of ends that divide $[\{E_k\}_k]$, ordered by division. If this set contains only $[\{E_k\}_k]$, then $[\{E_k\}_k]$ is a prime end.

Assume $\mathcal{E}$ has more than one element. Let $\mathcal{F}$ be a totally ordered subset of $\mathcal{E}$, indexed by a corresponding totally ordered set A. By the assumption given in the theorem, there is an element $[\{G_k\}_k]$ that divides all the elements of $\mathcal{F}$. Since each of these elements divides $[\{E_k\}_k]$ in turn, $[\{G_k\}_k]\Big|[\{E_k\}_k]$. Thus, $[\{G_k\}_k] \in \mathcal{E}$, satisfying the conditions for the use of Zorn's Lemma. Thus $\mathcal{E}$ has a minimal element, and this minimal element is necessarily a prime end. $\square$

Finally, we prove the following consequence of the assumptions made on Ω earlier in this section. This will be integral to our results in the next section.

Lemma 2.3.10. *Let Ω be a bounded domain satisfying the assumption given in Assumption 2.3.8, and let γ be a curve in Ω such that $\bigcap_{n\in\mathbb{N}} \overline{\gamma((n,\infty))} \subset \partial\Omega$. Then there is a prime end $[\{A_k\}_k]$ such that $[\{A_k\}_k] \Big| [\{F_k\}_k]$ for every $[\{F_k\}_k] \in E(\gamma)$, and $A_k \cap \gamma \neq \emptyset$ for every integer k.*

One cannot in general expect this prime end to be in $E(\gamma)$. Thus the best possible link the prime end has to γ is the condition that $A_k \cap \gamma \neq \emptyset$ for every integer k.

Proof. By Lemma 2.3.7, we know that $E(\gamma)$ has a minimal element $[\{G_k\}_k]$ and that $[\{G_k\}_k] \Big| [\{F_k\}_k]$ for every $[\{F_k\}_k] \in E(\gamma)$.

If $[\{G_k\}_k]$ happens to be a prime end, then set $[\{A_k\}_k] = [\{G_k\}_k]$ and the proof would be complete. If not, we may use Theorem 2.3.9 to obtain a prime end $[\{H_k\}_k]$ that divides $[\{G_k\}_k]$. Since $[\{G_k\}_k]$ is minimal in $E(\gamma)$, it must be the case that $[\{H_k\}_k] \notin E(\gamma)$. Now we have one of the two following possibilities:

(a) For every $k \in \mathbb{N}$, there is a positive real number t_k such that $\gamma(t_k) \in H_k$.

(b) There exists a positive integer k_0 such that $H_{k_0} \cap \gamma = \emptyset$.

If $[\{H_k\}_k]$ behaves as in (a), we simply take $[\{A_k\}_k] = [\{H_k\}_k]$ and the proof is complete. We now show that possibility (b) does not occur. We may also, without loss of generality, suppose that $\overline{H_k} \cap \Omega = H_k$.

Assume that $[\{H_k\}_k]$ behaves as in (b). For simplicity, we may take $k_0 = 1$. Then, define

$$m_H := \operatorname{dist}_M(\partial H_1 \cap \Omega, \partial H_2 \cap \Omega)$$

and

$$\widehat{H}_k := \left(\bigcup_{x\in H_2} B_M\left(x, \left(1-\frac{1}{k+1}\right)m_H\right)\right)\cap H_1.$$

Then $H_2 \subset \widehat{H}_k \subset \widehat{H}_{k+1} \subset H_1$ and

$$\operatorname{dist}_M(\partial\widehat{H}_k \cap \Omega, \partial\widehat{H}_{k+1}\cap\Omega) > 0$$

for every $k \in \mathbb{N}$. Because γ does not intersect H_1 and H_1 is relatively closed in Ω by assumption, we have that $\gamma \subset \Omega \setminus \overline{H_1}$.

Finally, we define D_k as the connected component of $G_k \setminus \overline{\widehat{H}_k}$ that contains the tail end of γ inside G_k. Since $\overline{\widehat{H}_k}$ contains no points of γ, we know that this component exists. By construction, $D_k \supset D_{k+1}$ and $\bigcap_{k\in\mathbb{N}} \overline{D_k} \subset \partial\Omega$. We need only show that $\operatorname{dist}_M(\partial D_k \cap \Omega, \partial D_{k+1}\cap\Omega) > 0$ for all $k \in \mathbb{N}$ to establish that $\{D_k\}_k$ is a chain. Let

$$M_k = \min\{\operatorname{dist}_M(\partial\widehat{H}_k\cap\Omega, \partial\widehat{H}_{k+1}\cap\Omega), \operatorname{dist}_M(\partial G_k\cap\Omega, \partial G_{k+1}\cap\Omega)\}.$$

Note that $M_k > 0$. Take $x \in \partial D_k \cap \Omega$ and $y \in \partial D_{k+1}\cap\Omega$ and consider the following cases.

Case 1: $x \in \partial G_k \cap \Omega$ and $y \in \partial G_{k+1}\cap\Omega$. In this case, we immediately have that $d_M(x,y) \geq M_k$.

Case 2: $x \in \partial G_k\cap\Omega$ and $y \in \partial\widehat{H}_{k+1}\cap\Omega$, but $y \notin \partial G_{k+1}\cap\Omega$. Here, it must be the case that $y \in G_{k+1}$. So $d_M(x,y) \geq M_k$.

Case 3: $x \in \partial\widehat{H}_k\cap\Omega$ and $y \in \partial\widehat{H}_{k+1}\cap\Omega$. As in Case 1, we immediately have that $d_M(x,y)\geq M_k$.

Case 4: $x \in \partial\widehat{H}_k\cap\Omega$ and $y \in \partial G_{k+1}\cap\Omega$, but $y \notin \partial\widehat{H}_{k+1}\cap\Omega$. Here, it must be that $y \in G_k \setminus \overline{\widehat{H}_{k+1}}$. So $d_M(x,y)\geq M_k$.

Case 5: $x \notin \Omega\cap(\partial G_k \cup \partial\widehat{H}_k)$ or $y \notin \Omega\cap(\partial G_{k+1}\cup\partial\widehat{H}_{k+1})$. We will focus on the first possibility, the second being handled in a very similar manner. Since $x \notin \Omega\cap(\partial G_k \cup$

$\partial\widehat{H}_k$), it follows that x is in the interior of G_k, and hence in the interior of $G_k \setminus \overline{\widehat{H}_k}$. It follows that for sufficiently small $r > 0$ the connected set $B_M(x, r) \subset G_k \setminus \overline{\widehat{H}_k}$, which then means that $B_M(x, r) \subset D_k$, violating the fact that $x \in \partial D_k$. Hence this case is not possible.

The above argument allows us to conclude that $\mathrm{dist}_M(\partial D_k \cap \Omega, \partial D_{k+1} \cap \Omega) > 0$, and so $\{D_k\}_k$ is a chain and $[\{D_k\}_k]$ is an end. By construction, $[\{D_k\}_k] \Big| [\{G_k\}_k]$ and $[\{D_k\}_k] \in E(\gamma)$. Because $[\{G_k\}_k]$ divides each end in $E(\gamma)$, and so $[\{D_k\}_k] = [\{G_k\}_k]$. However, by construction $[\{H_k\}_k]$ does not divide $[\{D_k\}_k]$, which violates the choice of $[\{H_k\}_k]$ as a prime end that divides $[\{G_k\}_k]$. Hence the alternative (b) cannot occur. This completes the proof of the lemma. □

2.4 Prime ends are from all sides.

The goal of this section is to show that if $V \subset \Omega$ is an open connected set such that $\partial V \cap \partial\Omega$ is non-empty, then there is a prime end in $\partial_P\Omega$ from the side of V. To do so we employ the inner metric dist^V_{inn} (see Definition 2.1.2). It should also be noted that we continue to have Assumption 2.3.8 hold for Ω.

For each $\varepsilon > 0$ let $V_\varepsilon := \{x \in V \,:\, \mathrm{dist}(x, X\setminus V) > \varepsilon\}$, and for a (locally rectifiable) curve γ in X, let

$$I(\gamma) := \bigcap_{n\in\mathbb{N}} \overline{\gamma((n,\infty))}.$$

Note that if $\gamma \subset \overline{V}$, then $I(\gamma)$ is a connected compact subset of $\overline{V}$.

Lemma 2.4.1. *Let $V \subset \Omega$ be an open connected set and suppose that $x_\infty \in \partial V \cap \partial\Omega$. Let $\{x_k\}_k$ be a sequence of points in V such that $\lim_k x_k = x_\infty$, and let $x_0 \in V$. Suppose that for each positive integer k the dist^V_{inn}-geodesic $\gamma^V_{x_0,x_k}$ does not intersect $\partial\Omega$. Then there is a curve $\gamma : [0,\infty) \to \overline{V}$ such that γ is the local uniform limit of a subsequence of the sequence of curves $\{\gamma^V_{x_0,x_k}\}$. Furthermore, if γ has infinite length, then $I(\gamma) \subset \partial V$.*

Remark 2.4.2. Note that if there are two points $z, w \in V$ such that the dist_{inn}^V-geodesic connecting z to w intersects $\partial\Omega$, then, because this geodesic has finite length (with respect to the metric d), it follows that there is a point $x_0 \in \partial V \cap \partial\Omega$ that is accessible from the side of V. See Definition 2.2.18 for the definition of "accessibility from the side" of V. As a consequence, if such points z, w exist, then there is a prime end from the side of V, and we can choose this prime end from the class $\partial_{SP}\Omega$.

Proof of Lemma 2.4.1. The existence of the curve γ is easily given by applying the Arzelà-Ascoli theorem to the equibounded (since Ω is bounded) equicontinuous (since these curves are 1-Lipschitz maps with respect to the underlying metric d) family $\{\gamma_{x_0,x_k}^V\}$. It is also clear that $I(\gamma) \subset \overline{V}$ and that each subcurve of γ is a dist_{inn}^V-geodesic between its endpoints. Demonstrating that, when γ has infinite length, $I(\gamma) \subset \partial V$ requires slightly more work.

We argue by contradiction. Suppose that there is a point $y \in I(\gamma) \cap V$, and pick a sequence $\{y_i\}$ with $y_i \to y$ and $y_i \in \gamma([i, \infty))$ for each i. Since V is open, there is a sufficiently small neighborhood of y within V such that the metrics d and dist_{inn}^V are biLipschitz equivalent inside this neighborhood. Thus, y_i converges to y with respect to dist_{inn}^V, requiring that $\mathrm{dist}_{inn}^V(y_i, y)$ be uniformly bounded by some $M < \infty$.

Since V is open and connected, $\mathrm{dist}_{inn}^V(x_0, y)$ must be finite; denote this quantity by N. By the triangle inequality,

$$\mathrm{dist}_{inn}^V(x_0, y_i) \leq \mathrm{dist}_{inn}^V(x_0, y) + \mathrm{dist}_{inn}^V(y_i, y) \leq N + M.$$

Since M and N are independent of i, we have that $\mathrm{dist}_{inn}^V(x_0, y_i)$ is uniformly bounded. But we picked y_i such that $y_i \in \gamma([i, \infty))$, and since γ is locally a geodesic with infinite length, $\mathrm{dist}_{inn}^V(x_0, y_i) \geq i$ for each i. But this contradicts the above bound on $\mathrm{dist}_{inn}^V(x_0, y_i)$. Thus, we have that $y \notin V$, that is, $I(\gamma) \subset \partial V$. □

Theorem 2.4.3. *Let Ω be a bounded connected open set satisfying the condition given* *in Assumption 2.3.8, and $V \subset \Omega$ be an open, connected set. If $\partial\Omega \cap \partial V$ is non-empty, then there is a prime end of Ω from the side of V.*

Proof. If there is a rectifiable curve in $\overline{V}$ that connects a point in V to a point in $\partial V \cap \partial\Omega$, then the accessibility results of [1] gives a corresponding prime end from the side of V. Indeed, if a curve γ connects a point in V to a point in $\partial V \cap \partial\Omega$, then the first time it intersects $\partial V \cap \partial\Omega$ will be a point that is accessible from Ω by that curve, and we can use that curve to construct a singleton prime end that is from the side of V. See also Remark 2.4.2 above.

Hence, without loss of generality, we may assume that there is no rectifiable curve in $\overline{V}$ that connects some point in V to a point in $\partial\Omega \cap \partial V$. Note that we now fulfill the assumptions of Lemma 2.4.1, allowing its use in the remainder of the proof.

We fix $x_0 \in V$ and $x_\infty \in \partial\Omega \cap \partial V$, and let $\{x_k\}_k$ be a sequence of points in V such that $\lim_k x_k = x_\infty$. For each positive integer k let $\gamma^V_{x_0,x_k}$ be as in the statement of Lemma 2.4.1. Clearly $\gamma^V_{x_0,x_k}$ cannot intersect $\partial\Omega$ because if it does, then we have a point in $\partial V \cap \partial\Omega$ that is accessible from V, violating the assumption stated in the previous paragraph of this proof. Hence Lemma 2.4.1 gives us a locally rectifiable curve $\gamma : [0,\infty) \to \overline{V}$ such that $\gamma(0) = x_0$, and for each $t > 0$ the curve $\gamma|_{[0,t]}$ is a $\operatorname{dist}^{\overline{V}}_{inn}$–geodesic that lies inside Ω. Since we assumed that there are no rectifiable curves connecting a point in V to $\partial V \cap \partial\Omega$, γ must have infinite length. Thus, $I(\gamma) \subset \partial V$.

Now the proof diverges according to two possibilities.

Case 1: $I(\gamma) \subset \partial\Omega \cap \partial V$. Then we can proceed to construct an end as follows. For $k \in \mathbb{N}$, we set E_k to be the connected component of $N(I(\gamma), 1/k) \cap \Omega$ that contains $\gamma([t_k,\infty))$ for some $t_k > 0$. Each E_k is an acceptable set, and $\{E_k\}_k$ satisfies the conditions of a chain. Note that

$$\operatorname{dist}_M(\Omega \cap \partial E_k, \Omega \cap \partial E_{k+1}) \geq \operatorname{dist}(\Omega \cap \partial E_k, \Omega \cap \partial E_{k+1}) \geq \tfrac{1}{k(k+1)} > 0$$

and that $I[\{E_k\}_k] = I(\gamma) \subset \partial\Omega \cap \partial V \subset \partial\Omega$.

It is clear that $[\{E_k\}_k]$ is from the side of V, however there is no *a priori* reason for $[\{E_k\}_k]$ to be prime. Note, however, that $[\{E_k\}_k]$ is clearly a member of $E(\gamma)$, and so by Lemma 2.3.10 there is a prime end $[\{A_k\}_k]$ dividing $[\{E_k\}_k]$ such that $A_k \cap \gamma \neq \emptyset$ for each k. Therefore, for each k, $A_k \cap \overline{V}$ must be nonempty. Furthermore, we can choose each A_k to be open (as mentioned in Remark 2.2.4), and so we conclude that $A_k \cap V$ is nonempty. It follows that $[\{A_k\}_k]$ is a prime end from the side of V.

Case 2: $I(\gamma) \not\subset \partial\Omega \cap \partial V$. We denote by $\overline{B}(x,r)$ the closed ball $\{y \in X | d(x,y) \leq r\}$ rather than the closure of the open ball $B(x,r)$. As justified in Remark 2.3.6, we may assume that X is geodesic. It follows that whenever $x \in X$ and $r > 0$, each pair of points $z, w \in \overline{B}(x,r)$ can be connected in the closed ball $\overline{B}(x,r)$ by a curve of length at most $2r$. Again, because X is doubling and hence separable, we can cover $\partial V \setminus \partial\Omega$ by at most a countable family of balls $B(z_i, r_i)$ with $z_i \in \partial V \setminus \partial\Omega$ and $r_i = \min\{\mathrm{dist}(z_i, X \setminus \Omega), d(z_i, x_0)\}/10$. Setting

$$V_j := V \cup \bigcup_{i=1}^{j} B(z_i, r_i)$$

for positive integers j, note that if $x, y \in V$, then

$$d(x,y) \leq \mathrm{dist}_{inn}^{V_{j+1}}(x,y) \leq \mathrm{dist}_{inn}^{V_j}(x,y) \leq \mathrm{dist}_{inn}^{V}(x,y). \tag{2.2}$$

As in the first part of the proof, we obtain curves γ_j for each j that are locally uniform limits of $\mathrm{dist}_{inn}^{V_j}$–geodesics connecting x_0 to x_k. Because of (2.2), and because each γ_n is a $\mathrm{dist}_{inn}^{V_n}$-geodesic, we know that if $\gamma_m(t_j) \in B(z_j, r_j) \cap V$ for some $m \leq j$, then for all $n \geq j$ we have that $\gamma_n([t_j + 2r_j, \infty))$ does not intersect $\overline{B}(z_j, r_j)$. It follows that for $n \geq j$ we have that $I(\gamma_n) \cap \overline{B}(z_j, r_j)$ is empty. Therefore,

$$I(\gamma_n) \subset \partial V \setminus \bigcup_{i=1}^{n} B(z_i, r_i).$$

A final application of Arzelà-Ascoli theorem gives a subsequence of $\{\gamma_n\}_n$ that converges locally uniformly to a curve $\beta : [0,\infty) \to \overline{\bigcup_j V_j}$ such that $\beta(0) = x_0$, and because for each $n \in \mathbb{N}$ we have that $\beta([t_n + 2r_n, \infty)) \cap \overline{B}(z_n, r_n)$ is empty,

$$I(\beta) \subset \partial V \setminus \bigcup_{i\in\mathbb{N}} B(z_i, r_i) = \partial V \cap \partial\Omega.$$

The proof is now completed by applying the argument at the end of the proof of Case 1 to β instead of γ. $\square$

The following corollary to the above theorem gives us a useful fact, namely that compact containment of connected sets in Ω is the same in both the Prime End topology and the topology on Ω inherited from X. Note however that if we do not require V to be connected, the following theorem would be false in general. Consider, for example, space $\hat{\Omega}$ used in the construction of Example 2.2.13. If we let define V as

$$\bigcup_{n=1}^{\infty} \left(\frac{1}{10}, \frac{1}{5}\right) \times \left(\frac{1}{2n - \frac{3}{2}}, \frac{1}{2n - \frac{1}{2}}\right),$$

then clearly $\overline{V} \cap \partial\Omega$ is nonempty, but $\overline{V}^{P,\Omega} \subset \Omega$.

Corollary 2.4.4. *Let $V \subset \Omega$ be an open, connected set. Then $\overline{V} \subset \Omega$ if and only if $\overline{V}^{P,\Omega} \subset \Omega$.*

Proof. If $\overline{V} \subset \Omega$, then $\overline{V} = \overline{V}^{P,\Omega}$. If $\overline{V}^{P,\Omega} \subset \Omega$, then clearly there can be no prime ends from the side of V. Thus, by Theorem 2.4.3, $\partial V \cap \partial\Omega = \emptyset$. Therefore, $\overline{V} \subset \Omega$. $\square$

CHAPTER 3

The Dirichlet Problem and the Perron Method

In this chapter, we introduce the necessary background material for the statement of the Dirichlet problem with prime end boundary data in a bounded domain of a doubling metric measure space. We then proceed to show that a solution to this problem can be constructed via the Perron method. Finally, we present some interesting examples exploring how the use of the Perron method with prime end boundary data can yield information about solutions to the classical Dirichlet problem.

3.1 Newton-Sobolev spaces and potential theory

We consider Newtonian spaces as the analog of Sobolev spaces in the metric setting.

Definition 3.1.1. Given a function $u : X \to [-\infty, \infty]$, we say that a non-negative Borel measurable function g on X is an *upper gradient* of u if whenever γ is a non-constant compact rectifiable curve in X, we have

$$|u(x) - u(y)| \leq \int\limits_{\gamma} g\, ds,$$

where x and y denote the two end points of γ. The above inequality should be interpreted to mean that $\int_\gamma g\, ds = \infty$ if at least one of $u(x)$, $u(y)$ is not finite.

Remark 3.1.2. Here, g is required to be Borel measurable to ensure that, for any curve γ, the composition $g \circ \gamma$ is measurable with respect to Lebesgue measure.

The notion of upper gradients is originally introduced by Heinonen and Koskela in [17], where it was called a very weak gradient. Clearly, if g is an upper gradient of u and ρ is a non-negative Borel measurable function on X, then $g+\rho$ is also an upper gradient of u. Thus, upper gradients are not unique. However, the collection of all upper gradients of u (if any exist) in $L^p(X)$ forms a convex subset of $L^p(X)$ and therefore, by the uniform convexity of $L^p(X)$ when $1 < p < \infty$, there is a unique function $g_u \in L^p(X)$ that is in the L^p-closure of this convex set, with minimal norm. Such a function g_u is called the *minimal p-weak upper gradient* of u.

Given $1 < p < \infty$, the Newtonian space $N^{1,p}(X)$ is the space

$$N^{1,p}(X) := \{u \in L^p(X) \mid u \text{ has an upper gradient } g \in L^p(X)\}/\sim,$$

where the equivalence relationship $\sim$ is such that $u \sim v$ if and only if

$$||u-v||_{N^{1,p}(X)} := \left[\int_X |u-v|^p \, d\mu + \inf_g \int_X g^p \, d\mu\right]^{1/p} = 0,$$

the infimum being taken over all upper gradients g of $u - v$.

While sets of measure zero perform admirably as exceptional sets for functions in $L^p(X)$, they are often too “large” to be considered exceptional sets for functions in $N^{1,p}(X)$. We consider sets of p-capacity zero as the appropriate exceptional sets for $N^{1,p}(X)$.

Definition 3.1.3. Given a set $A \subset X$, its *p-capacity* is the number

$$C_p(A : X) := \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ that satisfy $u \geq 1$ on A.

Definition 3.1.4. We say that X supports a p-Poincaré inequality if there are constants $C, \lambda \geq 1$ such that whenever u is a function on X with upper gradient g on X and B is a ball in X,

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq C \operatorname{rad}(B) \left(\frac{1}{\mu(\lambda B)} \int_{\lambda B} g^p \, d\mu \right)^{1/p}.$$

Here u_B denotes the integral average of u on B:

$$u_B := \frac{1}{\mu(B)} \int_B u \, d\mu.$$

Furthermore, we say that the measure μ on X is doubling if there is a constant $C \geq 1$ such that whenever B is a ball of radius $r > 0$ in X,

$$\mu(2B) \leq C \, \mu(B),$$

where $2B$ is a ball or radius $2r$.

As before, we will assume that μ is doubling and that X is complete and supports a p-Poincaré inequality. By results in [14], this implies that X is also quasiconvex.

Definition 3.1.5. Given a domain $\Omega \subset X$, the space of Newtonian functions with zero boundary values is the space

$$N_0^{1,p}(\Omega) := \{u \in N^{1,p}(X) \, : \, u = 0 \text{ in } X \setminus \Omega\}.$$

Given a function u defined only on Ω, we say that $u \in N_0^{1,p}(\Omega)$ if the zero-extension of u lies in $N_0^{1,p}(\Omega)$.

3.2 Prime end capacity and Newtonian spaces

In this section we will modify the notion of p-capacity introduced previously to take into consideration the structure of $\overline{\Omega}^P$. This new version of p-capacity was first introduced in [1] for the prime ends considered there.

Definition 3.2.1. For $E \subset \overline{\Omega}^P$ let

$$\overline{C}_p^P(E) = \inf_{u \in \mathcal{A}_E} ||u||^p_{N^{1,p}(\Omega)},$$

where $u \in \mathcal{A}_E$ if $u \in N^{1,p}(\Omega)$ satisfies both $u \geq 1$ on $E \cap \Omega$ and

$$\liminf_{\Omega \ni y \xrightarrow{\overline{\Omega}^P} x} u(y) \geq 1 \text{ for all } x \in E \cap \partial_P \Omega.$$

In the above definition, we can impose the additional requirement that $0 \leq u \leq 1$ without any change in the resulting number for E. This is because truncations of the form $T(u) = \min\{1, \max\{u, 0\}\}$ are norm-decreasing with respect to the $N^{1,p}(\Omega)$ norm.

The capacity $\overline{C}_p^P$ satisfies the usual basic properties of a capacity.

Lemma 3.2.2. *Let $E, E_1, E_2, E_3, \ldots$ be arbitrary subsets of $\overline{\Omega}$. Then*

1. $\overline{C}_p^P(\emptyset) = 0$,
2. $\mu(E \cap \Omega) \leq \overline{C}_p^P(E)$,
3. *If $E_1 \subset E_2$, then $\overline{C}_p^P(E_1) \leq \overline{C}_p^P(E_2)$ (monotonicity), and*
4. $\overline{C}_p^P\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \overline{C}_p^P(E_i)$ *(countable subadditivity).*

Proof. The first three properties are clear from the definition of $\overline{C}_p^P$. We now prove countable subadditivity. To do so, it suffices to assume that the sum on the right-hand side of the countable subadditivity inequality is finite.

Let $\varepsilon > 0$ and pick $u_i \in \mathcal{A}_{E_i}$ with upper gradient g_i in Ω such that

$$||u_i||^p_{L^p(\Omega)} + ||g_i||^p_{L^p(\Omega)} \leq \overline{C}_p^P(E_i) + \frac{\varepsilon}{2^i}.$$

Note that $0 \leq u_i \leq 1$. If we define $u = \sup_i u_i$ and $g = \sup_i g_i$, we see by [3, Lemma 1.28] that g is an upper gradient of u. By construction, $0 \leq u \leq 1$ and $u \in \mathcal{A}_E$, where $E = \bigcup_{i=1}^\infty E_i$. In particular, note that if $x \in E \cap \partial_P \Omega$, then there is some positive integer i_0 for which $x \in E_{i_0} \cap \partial_P \Omega$, and so by the fact that $u_{i_0} \in \mathcal{A}_{E_{i_0}}$,

$$\liminf_{\Omega \ni y \overset{\overline{\Omega}^P}{\to} x} u(y) \geq \liminf_{\Omega \ni y \overset{\overline{\Omega}^P}{\to} x} u_{i_0}(y) \geq 1.$$

Therefore

$$\overline{C}_p^P(E) \leq ||u||^p_{N^{1,p}(\Omega^P)} \leq \int_\Omega \sum_{i=1}^\infty u_i^p d\mu + \int_\Omega \sum_{i=1}^\infty g_i^p d\mu \leq \sum_{i=1}^\infty \overline{C}_p^P(E_i) + 2\varepsilon.$$

By letting $\varepsilon \to 0$, the proof is complete. $\square$

Definition 3.2.3. We say that a function u on $W \subset X$ is *p-quasicontinuous* (or, quasicontinuous) on W if for every $\varepsilon > 0$ we can find an open set $U_\varepsilon \subset X$ such that $u|_{W \setminus U_\varepsilon}$ is continuous and $C_p(U_\varepsilon) < \varepsilon$.

Proposition 3.2.4. *Suppose that the measure on X is doubling and supports a p-Poincaré inequality. Then $\overline{C}_p^P$ is an outer capacity, i.e. for all $E \subset \overline{\Omega}^P$,*

$$\overline{C}_p^P(E) = \inf_G \overline{C}_p^P(G),$$

where the infimum is taken over all $G \supset E$ that are open in $\overline{\Omega}^P$.

While the proof of this proposition is very similar to the proof of the related result [5, Proposition 3.3], the situation considered by [5] was simpler in that the boundary of the domain considered there was the Mazurkiewicz boundary, and so the function w defined there in a manner analogous to the proof here is easily seen to be admissible in computing the capacity. Here additional arguments were needed, and so we provide the complete proof here.

Proof. By the assumptions on X (doubling property of μ and the support of a p-Poincaré inequality) and by the results in [23] and [8] , we know that functions in $N^{1,p}(X)$ and functions in $N^{1,p}(\Omega)$ are p-quasicontinuous on X and Ω, respectively.

By the monotonicity of $\overline{C}_p^P$, we obtain the inequality $\overline{C}_p^P(E) \leq \inf_G \overline{C}_p^P(G)$ for free. We must work harder for the reverse inequality.

Given $E \subset \overline{\Omega}^P$ and $\varepsilon > 0$, we pick a function $u \in \mathcal{A}_E$ with $0 \leq u \leq 1$ such that

$$||u||_{N^{1,p}(\Omega)} \leq \overline{C}_p^P(E)^{1/p} + \varepsilon.$$

Since u is quasicontinuous on Ω, we may also take some open set $V \subset \Omega$ such that $C_p(V)^{1/p} \leq \varepsilon$ and $u|_{\Omega \setminus V}$ is continuous. Thus, $\{x \in \Omega | u(x) > 1 - \varepsilon\} \setminus V$ is an open set in $\Omega \setminus V$ with respect to the subspace topology. Therefore there is another open set $U \subset \Omega$ such that

$$U \setminus V = \{x \in \Omega | u(x) > 1 - \varepsilon\} \setminus V \supset (E \cap \Omega) \setminus V.$$

Because $C_p(V) \leq \varepsilon^p$, we can choose $v \in N^{1,p}(X)$ satisfying $||v||_{N^{1,p}(X)} < 2\varepsilon$, $0 \leq v \leq 1$ on X, and $v \geq 1$ on V. Set

$$w = \frac{u}{1-\varepsilon} + v.$$

Then $w \geq 1$ on $U \cup V$, which is an open set containing $E \cap \Omega$. Also, for each $[\{E_k\}_k] \in E \cap \partial_P \Omega$, there is a positive integer K such that $u > 1 - \varepsilon$ on E_K. Indeed, if not, then we can find a sequence of points $x_k \in E_k$ such that $u(x_k) \leq 1 - \varepsilon$ but

$x_k \overset{\overline{\Omega}^P}{\to} [\{E_k\}_k] \in E \cap \partial_P\Omega$, a violation of the choice of $u \in \mathcal{A}_E$.

Let

$$W = U \cup V \cup \bigcup_{[\{E_k\}_k] \in E \cap \partial_P \Omega} (E_K \cup E_K^P),$$

where E_K^P is as defined in Remark 2.2.12. Here, we have chosen the E_ks to be open. Then $W \supset E$ is an open set in $\overline{\Omega}^P$ and $w \in \mathcal{A}_W$. So

$$\begin{aligned}
\overline{C}_p^P(E)^{1/p} &\le \inf_G \overline{C}_p^P(G)^{1/p} \le \overline{C}_p^P(W)^{1/p} \le ||w||_{N^{1,p}(\Omega^P)} \\
&\le \frac{1}{1-\varepsilon}||u||_{N^{1,p}(\Omega^P)} + ||v||_{N^{1,p}(\Omega^P)} \le \frac{1}{1-\varepsilon}(\overline{C}_p^P(E)^{1/p} + \varepsilon) + 2\varepsilon.
\end{aligned}$$

By letting $\varepsilon \to 0$, the proof is complete. □

We also restate the definition of quasicontinuity with respect to this capacity.

Definition 3.2.5. A function $f : \overline{\Omega}^P \to \overline{\mathbb{R}}$ is $\overline{C}_p^P$*-quasicontinuous* if, for every $\varepsilon > 0$, there is a relatively open set $U \subset \overline{\Omega}^P$ such that $\overline{C}_p^P(U) < \varepsilon$ and $f|_{\overline{\Omega}^P \setminus U}$ is real-valued continuous.

It is natural for us to try to further relate $\overline{C}_p^P$ to the usual capacity C_p. To do this in a meaningful way, we would require a method of relating subsets of $\overline{\Omega}^P$ to those in $\overline{\Omega}$. Since single elements in $\overline{\Omega}^P$ might correspond to large sets in $\overline{\Omega}$, there is no easy mapping between $\overline{\Omega}^P$ and $\overline{\Omega}$ as in the case of the Mazurkiewicz boundary in [5]. Instead, we introduce the notion of the *Prime End Pushforward* of a set $E \subset \overline{\Omega}$ in the following way.

Definition 3.2.6. Given $E \subset X$, the Ω–*Prime End Pushforward* of E, denoted $P(E)$, is defined as

$$P(E) := (E \cap \Omega) \cup \{[\{E_k\}] \in \partial_P\Omega |\ \emptyset \neq I[\{E_k\}] \subset E\}.$$

It is clear from the definition that if $E \subset F$, then $P(E) \subset P(F)$. Also, this pushforward can be shown to be "an open map", in that if E is open in X, then $P(E)$ is open in $\overline{\Omega}^P$. In fact, if E is open, then

$$\{[\{E_k\}] \in \partial_P\Omega |\ I[\{E_k\}] \subset E\} = \{[\{E_k\}_k] \in \partial_P\Omega\ |E_j \subset G \text{ for some } j\},$$

and thus $P(E) = (E \cap \Omega)^P$.

If E is open and $I[\{E_k\}_k] \subset E$ then for some $\varepsilon > 0$, we have that $N(I[\{E_k\}], \varepsilon) \subset E$. Then, by Lemma 2.3.1, there is a j such that $E_j \subset E$. Conversely, if there is a j such that $E_j \subset E$, then $\bigcap_{k=j}^{\infty} E_k \subset E$, and so $I[\{E_k\}_k] \subset E$. Hence, if $E \subset \overline{\Omega}$ is relatively open, then $P(E)$ is open in $\overline{\Omega}^P$.

With this definition, we have the following Lemma. Recall that we assume the measure on X to be doubling and support a p-Poincaré inequality.

Lemma 3.2.7. *Let $E \subset X$. Then*

$$\overline{C}_p^P(P(E)) \leq C_p(E).$$

Proof. Given any $\varepsilon > 0$, we may pick an open set $G \supset E$ in X such that $C_p(G) \leq C_p(E) + \varepsilon/2$. This is due to the fact that C_p is an outer capacity (see [8, Corollary 1.3] or [3, Theorem 5.21]).

Let $f \in N^{1,p}(X)$ such that $f = 1$ on G and $||f||^p_{N^{1,p}(X)} < C_p(G) + \varepsilon/2$. Define $\tilde{f} := f|_\Omega$. Note that $\tilde{f} \in N^{1,p}(\Omega)$.

Immediately, if $x \in P(G) \cap \Omega = G \cap \Omega$, then $\tilde{f}(x) = 1$. For $x \in P(G) \cap \partial_P\Omega$, we must look at a sequence $\{y_k\}_k$ in Ω converging to x in $\overline{\Omega}^P$. Since $P(G)$ is open, we may assume that $y_k \in P(G) \cap \Omega$ for each k. Then, clearly, $\liminf_{y_k \xrightarrow{\overline{\Omega}^P} x} \tilde{f}(y_k) \geq 1$. Thus, $\tilde{f}$

is an admissible function for the computation of $\overline{C}_p^P(P(G))$. So,

$$\overline{C}_p^P(P(E)) \leq \overline{C}_p^P(P(G)) \leq ||\tilde{f}||^p_{N^{1,p}(\Omega)} \leq C_p(E) + \varepsilon.$$

By letting $\varepsilon \to 0$, the proof is completed. $\square$

Finally, in order to compare boundary values of functions on $\overline{\Omega}^P$, we need to consider $N_0^{1,p}(\Omega)$ as given in Definition 3.1.5.

The following proposition is analogous to Proposition 5.4 of [5]. The major difference between our situation here and that of [5] is that there is no continuous map $\Phi : \partial_P\Omega \to \partial\Omega$, and so the proof of the following proposition is more complicated than that found in [5].

Proposition 3.2.8. *If $f \in N_0^{1,p}(\Omega)$, then the zero-extension of f to $\partial_P\Omega$ is $\overline{C}_p^P$-quasicontinuous.*

Proof. Let f^0 be the zero extension of f (as a function on Ω) to all of X. Then $f^0 \in N^{1,p}(X)$, and so for any $\varepsilon > 0$ there is an open set U_ε in X such that $C_p(U_\varepsilon) < \varepsilon$ and $f^0|_{X\setminus U_\varepsilon}$ is continuous. Now let $\hat{f} : \overline{\Omega}^P \to \overline{\mathbb{R}}$ be defined as

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \partial_P\Omega. \end{cases}$$

By Lemma 3.2.7 we know that $\overline{C}_p^P(P(U_\varepsilon)) < \varepsilon$. We wish to show that $\hat{f}|_{\overline{\Omega}^P\setminus P(U_\varepsilon)}$ is continuous. Let $x \in \overline{\Omega}^P \setminus P(U_\varepsilon)$ and $\{y_k\}_k$ be a sequence in $\overline{\Omega}^P \setminus P(U_\varepsilon)$ such that $y_k \overset{\overline{\Omega}^P}{\to} x$. We wish to check that $\hat{f}(y_k) \to \hat{f}(x)$. Since $\hat{f}|_{\Omega\setminus P(U_\varepsilon)} = f|_{\Omega\setminus U_\varepsilon}$ is continuous, we know that if $x \in \Omega$ then the above convergence holds. So without loss of generality, we may consider the following two cases.

Case 1: $y_k \in \partial_P\Omega$ for each k, and $x \in \partial_P\Omega$. Since $\hat{f}(y_k) = 0 = \hat{f}(x)$ for all k, this case is immediate.

Case 2: $y_k \in \Omega$ for each k, and $x \in \partial_P \Omega$. Let $\{y_{k_i}\}_i$ be a subsequence of $\{y_k\}_k$. Since $I[x]$ is a compact set and $\overline{\Omega}$ is a compact subset of X, there is a further subsequence $\{y_{k_{i,j}}\}_j$ such that, for some $x_0 \in I[x]$, $y_{k_{i,j}} \to x_0$ in the topology of X. Since $y_{k_{i,j}} \in X \setminus U_\varepsilon$ for each j and $X \setminus U_\varepsilon$ is a closed set, the limit x_0 of $y_{k_{i,j}}$ cannot lie in U_ε. Therefore $x_0 \in X \setminus (\Omega \cup U_\varepsilon)$, $f^0(x_0) = 0$ and $\hat{f}(y_{k_{i,j}}) = f^0(y_{k_{i,j}}) \to 0$. Since this happens for all subsequences of $\{y_k\}_k$, we conclude that $\hat{f}(y_k) \to 0 = \hat{f}(x)$.

With both possibilities dealt with, we have proved the desired claim. $\square$

It is possible to define a dual notion to the *Prime End Pushforward* of set, namely the *Prime End Pullback*.

Definition 3.2.9. Given $F \subset \overline{\Omega}^P$, the Ω–*Prime End Pullback* of F, denoted $P^{-1}(F)$, is defined as

$$P^{-1}(F) := (F \cap \Omega) \cup \bigcup_{[\{E_k\}] \in F} I[\{E_k\}].$$

It is natural to consider how the two notions interact. Given an $E \subset X$ and $F \subset \overline{\Omega}^P$, then

$$\begin{aligned} P^{-1}(P(E)) &= P^{-1}\left((E \cap \Omega) \cup \{[\{E_k\}] \in \partial_P \Omega |\ I[\{E_k\}] \subset E\}\right) \\ &= (E \cap \Omega) \cup \bigcup_{I[\{E_k\}] \subset E} I[\{E_k\}] \\ &\subset E \end{aligned}$$

and

$$\begin{aligned} P(P^{-1}(F)) &= P\left((F \cap \Omega) \cup \bigcup_{[\{E_k\}] \in F} I[\{E_k\}]\right) \\ &= (F \cap \Omega) \cup \{[\{E_k\}] \in \partial_P \Omega |\ I[\{E_k\}_k] \subset \bigcup_{[\{E_k\}] \in F} I[\{E_k\}]\} \qquad \supset F \end{aligned}$$

As the following two examples show, equality does not hold in general for either case.

Example 3.2.10. If we take $X = \mathbb{R}^2$, with

$$\Omega := (0,1)^2 \setminus \bigcup_{n=2}^{\infty} \{1/n\} \times (0, 1/2],$$

and let $E = [0,1]^2$, we observe that

$$P^{-1}(P(E)) = [0,1]^2 \setminus \{0\} \times [0, 1/2).$$

Thus, $E \not\subset P^{-1}(P(E))$ in this case.

Example 3.2.11. Letting $X = \mathbb{R}^2$ and let Ω be the slit disk

$$\Omega = B(0,1) \setminus [0,1) \times \{0\}.$$

Take (recalling Remark 2.2.12)

$$F = \{(x,y) \in \Omega \mid y > 0\}^P.$$

Then $F \subset \overline{\Omega}^P$ consists of the upper half of the slit disk in addition to the prime ends associated with the 'top' part of the slit. It is then easy to see that $P(P^{-1}(F))$ will contain both 'sides' of the slit, and so $P(P^{-1}(F)) \not\subset F$.

We also state a companion Lemma to Lemma 3.2.7.

Lemma 3.2.12. *Given $F \subset \overline{\Omega}^P$, we have*

$$\overline{C}_p^P(F) \leq C_p(P^{-1}(F)).$$

Proof. By Lemma 3.2.7, $\overline{C}_p^P(P(P^{-1}(F))) \leq C_p(P^{-1}(F))$. Since $F \subset P(P^{-1}(F))$, we immediately have that $\overline{C}_p^P(F) \leq C_p(P^{-1}(F))$, completing the proof. $\square$

3.3 The Perron solution with respect to Prime Ends

In this section we employ the Perron method to construct a solution to the Dirichlet problem with prime end boundary data. The classical Perron method was introduced in [22] in 1923, and has been studied extensively in Euclidean domains. The Perron method is useful in that it allows us to construct a solution to the Dirichlet problem without requiring our boundary data to be continuous. Recently, the Perron method was used successfully to construction solutions to the Dirichlet problem for bounded domains in doubling metric measure spaces, see [7].

Before stating the Dirichlet problem and defining its Perron solution, we must make several definitions.

Definition 3.3.1. A function $u \in N^{1,p}(\Omega)$ is said to be a *p-minimizer in* Ω if it has minimal p-energy in Ω. That is, for all $\phi \in N_0^{1,p}(\Omega)$,

$$\int_{\operatorname{supp}(\phi)} g_u^p \, d\mu \leq \int_{\operatorname{supp}(\phi)} g_{u+\phi}^p \, d\mu.$$

Here, g_u and $g_{u+\phi}$ denote the minimal p-weak upper gradient of u and $u + \phi$, respectively. A function that satisfies this condition for all nonnegative $\phi \in N_0^{1,p}(\Omega)$ is said to be a *p-superminimizer in* Ω. A function is said to be *p-harmonic in* Ω if it is a continuous p-minimizer in Ω.

As the results in [19] show that, under the hypotheses considered here, every p-minimizer can be modified on a set of p-capacity zero to obtain a locally Hölder continuous p-harmonic function.

The lower semicontinuous regularization of a function u is

$$u^*(x) = \operatorname{ess} \liminf_{y \to x} u(y).$$

As shown in [18], the equality $u^* = u$ holds outside a set of zero p-capacity when

u is a p-superminimizer. For this reason, any p-superminimizer discussed here will be assumed to be lower semicontinuously regularized in this manner. Therefore, if u is a p-superminimizer, then for every real number t the set $\{x \in \Omega | u(x) > t\}$ is open. Recall from Definition 3.1.5 above that a function defined on Ω is in $N_0^{1,p}(\Omega)$ if its zero-extension to $X \setminus \Omega$ is in $N^{1,p}(X)$.

Definition 3.3.2. Let $V \subset X$ be open and bounded, with $C_p(X \setminus V) > 0$. Then, for $f \in N^{1,p}(V)$ and $\psi : V \to \overline{\mathbb{R}}$, we define the set

$$\mathcal{K}_{\psi,f}(V) := \{v \in N^{1,p}(V) | v - f \in N_0^{1,p}(V),\ v \geq \psi \text{ a.e. in } V\}.$$

A function $u \in \mathcal{K}_{\psi,f}(V)$ is said to be a *solution of the* $\mathcal{K}_{\psi,f}(V)$*-obstacle problem* if

$$\int_V g_u^p d\mu \leq \int_V g_v^p d\mu, \text{ for all } v \in \mathcal{K}_{\psi,f}(V).$$

It is shown in [18, Theorem 3.2] that solutions to the $\mathcal{K}_{\psi,f}(V)$-obstacle problem exist and are unique (in $N^{1,p}(V)$), provided $\mathcal{K}_{\psi,f}(V) \neq \emptyset$.

Given a function $f \in N^{1,p}(\Omega)$ and a bounded domain $\Omega \subset X$ with $C_p(X \setminus \Omega) > 0$, it is a result of [13, Theorem 2.7] that if $\mathcal{K}_{\psi,f}(\Omega) \neq \emptyset$, then there is a unique function $u \in N^{1,p}(\Omega)$ such that u is the solution to the $\mathcal{K}_{\psi,f}(\Omega)$-obstacle problem. We are particularly interested in the application of this result to the $\mathcal{K}_{-\infty,f}$-obstacle problem. We shall refer to the solution of this problem as $H_\Omega f$, though we will often suppress the subscript and simply refer to it as Hf. The condition $C_p(X \setminus \Omega) > 0$ is needed in order to have non-trivial solutions in Ω. Should $C_p(X \setminus \Omega) = 0$, then $N_0^{1,p}(\Omega) = N^{1,p}(X)$, and in this case for every non-negative $f \in N_0^{1,p}(\Omega)$ we would have that $H_\Omega f$ is a non-negative p-harmonic function on X itself, and hence by the Harnack indequality (see [19]) we would have $H_\Omega f = 0$. By assuming $C_p(X \setminus \Omega) > 0$, we avoid this problem.

Our setting in this paper will primarily be $\overline{\Omega}^P$. In general $\overline{\Omega}^P$ may not be metrizable, and thus the notion of $N^{1,p}(\overline{\Omega}^P)$ may not make sense. However, since the subspace topology of Ω inherited from $\overline{\Omega}^P$ agrees with the standard metric topology on Ω inherited from X, the Newton-Sobolev space $N^{1,p}(\Omega)$ can be seen as the function space corresponding to both Ω, seen as a domain in X, and Ω, seen as a domain in $\overline{\Omega}^P$.

However, one should keep in mind that in general functions in $N^{1,p}(X)$, when restricted to Ω, may not have a natural extension to $\partial_P\Omega$.

Now we are ready to consider the following Dirichlet problem: Given $g : \partial_P\Omega \to \mathbb{R}$, find a function u that is p-harmonic on Ω and such that $u = g$ on $\partial_P\Omega$ in some sense. The method we use to construct possible solutions to the above problem for certain type of functions g is the Perron method, adapted to $\partial_P\Omega$. We continue to assume the standard assumptions about X (the doubling property of the measure on X, and the validity of a p-Poincaré inequality on X), and that Ω is a bounded domain in X with $C_p(X \setminus \Omega) > 0$ such that Ω satisfies the condition given in Assumption 2.3.8.

Definition 3.3.3. A function $u\colon \Omega \to (-\infty, \infty]$ is said to be *p-superharmonic* if

1. u is lower semicontinuous,

2. u is not identically ∞ on Ω, and

3. for every nonempty open set $V \Subset \Omega$ and all functions $v \in Lip(X)$, if $v \leq u$ on ∂V, then $H_V v \leq u$ in V.

A function u is said to be *p-subharmonic* if $-u$ is p-superharmonic.

We now construct the Perron solution with respect to $\overline{\Omega}^P$.

Definition 3.3.4. Given a function $f\colon \partial_P\Omega \to \overline{\mathbb{R}}$, let $\mathcal{U}_f(\overline{\Omega}^P)$ be the set of all p-

superharmonic functions u on Ω bounded below such that

$$\liminf_{\Omega \ni y \overset{\overline{\Omega}^P}{\to} [\{E_n\}_n]} u(y) \geq f([\{E_n\}_n]) \text{ for all } [\{E_n\}_n] \in \partial_P \Omega.$$

We define the *upper Perron solution* of f by

$$\overline{P}_{\overline{\Omega}^P} f(x) = \inf_{u \in \mathcal{U}_f(\overline{\Omega}^P)} u(x), \ x \in \Omega.$$

Similarly, let $\mathcal{L}_f(\overline{\Omega}^P)$ be the set of all p-subharmonic functions u on Ω bounded above such that

$$\limsup_{\Omega \ni y \overset{\overline{\Omega}^P}{\to} [\{E_n\}_n]} u(y) \leq f([\{E_n\}_n]) \text{ for all } [\{E_n\}_n] \in \partial_P \Omega.$$

We define the *lower Perron solution* of f by

$$\underline{P}_{\overline{\Omega}^P} f(x) = \sup_{u \in \mathcal{L}_f(\overline{\Omega}^P)} u(x), \ x \in \Omega^P.$$

Note that $\underline{P}_{\overline{\Omega}^P} f = -\overline{P}_{\overline{\Omega}^P}(-f)$. If $\overline{P}_{\overline{\Omega}^P} f = \underline{P}_{\overline{\Omega}^P} f$ on Ω, then we let $P_{\overline{\Omega}^P} f := \overline{P}_{\overline{\Omega}^P} f$, and f is said to be *resolutive*.

For the classical formulation of the Perron solution, it is shown in [7, Theorem 5.1] that functions $f \in N^{1,p}(X)$ are resolutive. We wish to provide a similar result for an appropriate class of functions on $\partial_P \Omega$. Due to the potential non-compactness of the space $\overline{\Omega}^P$, we must first prove that several important results still hold in this space. Chief among them is the following comparison principle. An analogous comparison principle, set up for the Mazurkiewicz boundary in [5, Proposition 7.2], is straightforward to prove because of the assumption in [5] that the Mazurkiewicz boundary $\partial_M \Omega$ is compact. Here we overcome the lack of compactness of $\partial_P \Omega$ with the aid of Corollary 2.4.4.

Proposition 3.3.5. *Assume that u is p-superharmonic and that v is p-subharmonic in Ω. If*

$$\infty \neq \limsup_{\Omega \ni y \overset{\overline{\Omega}^P}{\to} x} v(y) \leq \liminf_{\Omega \ni y \overset{\overline{\Omega}^P}{\to} x} u(y) \neq -\infty \text{ for each } x \in \partial_P \Omega,$$

then $v \leq u$ in Ω.

Proof. Let $\varepsilon > 0$. Since u is lower semicontinuous and v is upper semicontinuous, we know that $V_\varepsilon := \{y \in \Omega : u(y) - v(y) > -\varepsilon\}$ is an open subset of Ω.

By the assumption of this proposition, for each $x \in \partial_P \Omega$ we can find a neighborhood V_ε^x of x in $\overline{\Omega}^P$ such that $v < u + \varepsilon$ in $V_\varepsilon^x \cap \Omega$. Note that $V_\varepsilon^x \cap \Omega \subset V_\varepsilon$ for each $x \in \partial_P \Omega$. Thus $V_\varepsilon \cup \partial_P \Omega$ is an open subset of $\overline{\Omega}^P$.

Let $U_\varepsilon = \overline{\Omega}^P \setminus \overline{V_\varepsilon}$ and C_ε be a connected component of U_ε. Then, by Lemma 2.3.4, $\overline{C_\varepsilon}^{P,\Omega} \subset \Omega$ and $v \leq u + \varepsilon$ on $\partial_P^\Omega C_\varepsilon$.

By Corollary 2.4.4, we know that $\overline{C_\varepsilon} \subset \Omega$, and, since $\partial_P^\Omega C_\varepsilon = \partial C_\varepsilon$ in this case, $v \leq u + \varepsilon$ on ∂C_ε. We now proceed as in the proof of [18, Theorem 7.2] (the standard comparison theorem for p-super/sub harmonic functions) to see that $v \leq u + \varepsilon$ in C_ε. Since this inequality holds for each connected component of U_ε, we conclude that $v \leq u + \varepsilon$ in U_ε. By letting $\varepsilon \to 0$, the proof is complete. $\square$

An immediate consequence of Proposition 3.3.5 is the following Corollary.

Corollary 3.3.6. *If $f \colon \partial_P \Omega \to \mathbb{R}$, then*

$$\underline{P}_{\overline{\Omega}^P} f \leq \overline{P}_{\overline{\Omega}^P} f.$$

Lemma 3.3.7. *Let $\{U_k\}_{k=1}^\infty$ be a decreasing sequence of relatively open sets in $\overline{\Omega}^P$ such that $\overline{C}_p^P(U_k) < 2^{-kp}$. Then there exists a decreasing sequence of nonnegative functions $\{\psi_j\}_{j=1}^\infty$ on Ω such that $||\psi_j||_{N^{1,p}(\Omega)} < 2^{-j}$, $\psi_j \geq k - j$ in $U_k \cap \Omega$, and $\lim_{\Omega \ni y \overset{\overline{\Omega}^P}{\to} x} \psi_j(y) \geq k - j$ for all $x \in U_k \cap \partial_P \Omega$.*

Proof. Let f_k with $||f_k||_{N^{1,p}(\Omega)} < 2^{-k}$ be admissible functions for the computation of $\overline{C}_p^P(U_k)$. Then $\psi_j = \sum_{k=j+1}^{\infty} f_k$ has the properties required. $\square$

Before we state our main theorem, we first need the following two results.

Proposition 3.3.8. *Let $\{f_j\}_{j=1}^{\infty}$ be a p-quasieverywhere decreasing sequence of functions in $N^{1,p}(\Omega)$ such that $f_j \to f$ in $N^{1,p}(\Omega)$ as $j \to \infty$. Then Hf_j decreases to Hf locally uniformly in Ω.*

If u and u_j are solutions to the $\mathcal{K}_{f,f}$ and $\mathcal{K}_{f_j,f_j}$-obstacle problems, then $\{u_j\}_{j=1}^{\infty}$ decreases q.e. in Ω to u.

Lemma 3.3.9. *For every function $f : \partial\Omega \to \overline{\mathbb{R}}$, the upper Perron solution $\overline{P}f$ is p-harmonic in Ω or is identically $\pm\infty$.*

The first paragraph of Proposition 3.3.8 is a slight restatement of [24, Corollary 4.8], reproduced below.

Corollary 3.3.10. *Let Ω be a bounded domain in a proper metric space X that is equipped with a doubling measure supporting a weak $(1,p)$-Poincaré inequality, and let u_i be a sequence of functions that are p-harmonic in Ω and are bounded in $N^{1,p}_{loc}(X)$. Then there is a subsequence u_{i_k} converging to an $N^{1,p}_{loc}(X)$-function u lcoally uniformly in Ω, and furthermore, u is p-harmonic.*

In Proposition 3.3.8, we simply restrict ourselves to sequences in $N^{1,p}(\Omega) \subset N^{1,p}_{loc}(X)$.

The second paragraph of Proposition 3.3.8 is proved in [12, Theorem 3.1]. Farnana considers in her paper the double obstacle problem; that is, she has an extended real-valued function which bounds possible solutions from above in addition to the lower obstacle we consider here. However, by simply taking the upper obstacle to be identically infinity, we see that our obstacle problem is merely a special case of hers.

Lemma 3.3.9 appears in [7] as Theorem 4.1. To aid the flow of exposition here, we will not reproduce the proof of Lemma 3.3.9 here. However, for the sake of completeness, we will briefly outline the idea of the proof. The proof relies on taking the *Poisson modification* of functions in $\mathcal{U}_f$ in order to ensure that they are p-harmonic on larger and larger compactly contained subdomains of Ω. The Poisson modification taken here is for nonregular domains, and is described in the lemma below.

Lemma 3.3.11. *Let $\Omega' \Subset \Omega$ be a subdomain and u be a p-superharmonic function in Ω locally bounded from above. Let*

$$u' = \begin{cases} u(x), & if\ x \in \Omega \setminus \Omega' \\ H_{\Omega'}u(x), & if\ x \in \Omega' \\ \min\{u(x),\ \liminf_{\Omega' \ni y \to x} H_{\Omega'}u(y)\}, & if\ x \in \partial\Omega'. \end{cases}$$

Then u' is p-superharmonic in Ω and $u' \leq u$ in Ω.

This lemma is a reproduction of [7, Lemma 4.2]. The proof given there also works here, as under Assumption 2.3.8 compactly contained subdomains are equivalent under the prime end topology and the metric topology.

We now state the main theorem of this paper.

Theorem 3.3.12. *Let $f : \overline{\Omega}^P \to \overline{R}$ be a $\overline{C}_p^P$-quasicontinuous function such that $f|_\Omega$ is in $N^{1,p}(\Omega)$. Then f is resolutive and $P_{\overline{\Omega}^P} f = Hf|_\Omega$.*

By having overcome the drawback from the fact that $\partial_P\Omega$ may not be compact with the help of Proposition 3.3.5, the proof of the above main theorem is very similar to that of [5, Theorem 7.4]. However, one difference still remains: namely the topology of $\overline{\Omega}^P$ near the boundary $\partial_P\Omega$, which is not as simple as that of the Mazurkiewicz boundary.

Proof. To avoid excessive subscripts, we will refer to $Hf|_\Omega$ simply as Hf.

First, we assume that $f \geq 0$. We extend Hf to $\overline{\Omega}^P$ by letting $Hf = f$ on $\partial_P\Omega$. We now show that this extension is $\overline{C}_p^P$-quasicontinuous.

Let $h = f - Hf$. Then, since Hf is a solution to the $\mathcal{K}_{-\infty,f}(\Omega)$-obstacle problem, $h \in N_0^{1,p}(\Omega)$. Additionally, h is quasicontinuous on Ω with $\overline{C}_p^P$-quasicontinuous extension $h = 0$ to $\partial_P\Omega$, see Proposition 3.2.8. Because f is $\overline{C}_p^P$-quasicontinuous on $\overline{\Omega}^P$, it now follows that so is Hf.

Pick open sets $\{G_j\}$ in $\overline{\Omega}^P$ with $\overline{C}_p^P(G_j) < 2^{-jp}$ such that $Hf|_{\overline{\Omega}^P \setminus G_j}$ is continuous. Defining $U_k = \bigcup_{j=k+1}^{\infty} G_j$, we see that $\overline{C}_p^P(U_k) < 2^{-kp}$ and $Hf|_{\overline{\Omega}^P \setminus U_k}$ is still continuous.

These sets $\{U_k\}$ fulfill the conditions of Lemma 3.3.7, and so we may take functions $\{\psi_j\}$ as described in that Lemma. We set $f_j = Hf + \psi_j$ (note here that f_j is a function on Ω alone) and let ϕ_j be the lower semicontinuously regularized solution of the $\mathcal{K}_{f_j,f_j}(\Omega)$-obstacle problem as given in Definition 3.3.2.

For each positive integer m we have that

$$f_j \geq \psi_j \geq m \text{ on } U_{m+j} \cap \Omega.$$

Given $\varepsilon > 0$, let $x \in \partial_P\Omega$. If $x \notin U_{m+j}$, by the continuity of $Hf|_{\overline{\Omega}^P \setminus U_{m+j}}$, there is a neighborhood V_x of x in $\overline{\Omega}^P$ such that

$$f_j(y) \geq Hf(y) \geq Hf(x) - \varepsilon = f(x) - \varepsilon \text{ for all } y \in (V_x \cap \Omega) \setminus U_{m+j}.$$

So, if $x \in \partial_P\Omega \setminus U_{m+j}$,

$$f_j \geq \min\{f(x) - \varepsilon, m\} \text{ in } V_x \cap \Omega^P.$$

If, instead, $x \in U_{m+j}$, we take $V_x = U_{m+j}$.

Now, by the previous paragraphs,

$$f_j \geq \min\{f(x) - \varepsilon, m\} \text{ in } V_x \cap \Omega.$$

Since ϕ_j is the solution of the $\mathcal{K}_{f_j,f_j}$-obstacle problem, $\phi_j \geq f_j$ quasieverywhere. Therefore, $\phi_j(y) \geq \min\{f(x)-\varepsilon, m\}$ quasieverywhere in $V_x \cap \Omega$. However, ϕ_j is lower semicontinuously regularized, and hence $\phi_j \geq f_j$ everywhere. Thus, $\phi_j(y) \geq \min\{f(x)-\varepsilon, m\}$ for all $y \in V_x \cap \Omega$. Therefore,

$$\liminf_{\Omega \ni y \overset{\overline{\Omega}^P}{\to} x} \phi_j(y) \geq \min\{f(x)-\varepsilon, m\}.$$

As $\varepsilon \to 0$ and $m \to \infty$, we have that

$$\liminf_{\Omega \ni y \overset{\overline{\Omega}^P}{\to} x} \phi_j(y) \geq f(x) \text{ for all } x \in \partial_p \Omega.$$

Since ϕ_j is p-superharmonic, we have that $\phi_j \in \mathcal{U}_f(\overline{\Omega}^P)$, and so $\phi_j \geq \overline{P}_{\overline{\Omega}^P} f$. Because Hf is the solution to the $\mathcal{K}_{Hf,Hf}(\Omega)$-obstacle problem, by Proposition 3.3.8 we know that ϕ_j decreases quasieverywhere to Hf, that is, $\overline{P}_{\overline{\Omega}^P} f \leq Hf$ q.e. in Ω when $f \geq 0$.

Note that if $f \in N^{1,p}(\Omega)$ has a $\overline{C}_p^P$-quasicontinuous extension to $\overline{\Omega}^P$, then so does $\max\{f, m\}$ for each integer m. Therefore, for $f \in N^{1,p}(\Omega)$, not necessarily non-negative,

$$\overline{P}_{\overline{\Omega}^P} f \leq \lim_{m \to -\infty} \overline{P}_{\overline{\Omega}^P} \max\{f, m\} \leq \lim_{m \to \infty} H \max\{f, m\} = Hf \quad \text{q.e. in } \Omega.$$

Because $\overline{P}_{\overline{\Omega}^P} f$ is p-harmonic in Ω and hence is continuous, we have that both $\overline{P}_{\overline{\Omega}^P} f$ and Hf are continuous. Therefore $\overline{P}_{\overline{\Omega}^P} f \leq Hf$ everywhere in Ω.

Finally, with the aid of Proposition 3.3.5, or more precisely, with the help of Corollary 3.3.6, we see that

$$\underline{P}_{\overline{\Omega}^P} f = -\overline{P}_{\overline{\Omega}^P}(-f) \geq -H(-f) = Hf \geq \overline{P}_{\overline{\Omega}^P} f \geq \underline{P}_{\overline{\Omega}^P} f.$$

Thus $Hf = \underline{P}_{\overline{\Omega}^P} f = \overline{P}_{\overline{\Omega}^P} f$ and f is resolutive. $\square$

The following results show that the solution $P_{\overline{\Omega}^P} f$ is stable under perturbation of f on a set of $\overline{C}_p^P$ capacity zero.

Proposition 3.3.13. *Let $f\colon \overline{\Omega}^P \to \overline{\mathbb{R}}$ be a $\overline{C}_p^P$-quasicontinuous function with $f|_\Omega$ in the class $N^{1,p}(\Omega)$. If $h\colon \partial_P\Omega \to \overline{\mathbb{R}}$ is zero $\overline{C}_p^P$ quasi-everywhere, then $f+h$ is resolutive with respect to $\overline{\Omega}^P$, and $P_{\overline{\Omega}^P}(f+h) = P_{\overline{\Omega}^P}(f)$.*

Proof. We may extend h into Ω by zero, and clearly $h|_\Omega \in N^{1,p}(\Omega)$. Note that, since $\overline{C}_p^P$ is an outer capacity (see Lemma 3.2.4), this extended function h is $\overline{C}_p^P$-quasicontinuous. Thus $f+h$ is $\overline{C}_p^P$-quasicontinuous. Finally, $(f+h)|_\Omega \in N^{1,p}(\Omega)$, so by using Theorem 3.3.12, $f+h$ is resolutive and $P_{\overline{\Omega}^P}(f+h) = H(f+h)$. Since $f = f+h$ in Ω, we therefore have $Hf = H(f+h)$. Thus, by Theorem 3.3.12 again,

$$P_{\overline{\Omega}^P}(f+h) = H(f+h) = Hf = P_{\overline{\Omega}^P} f.$$

$\square$

Corollary 3.3.14. *Let $f\colon \overline{\Omega}^P \to \overline{\mathbb{R}}$ be a bounded $\overline{C}_p^P$-quasicontinuous function with $f|_\Omega \in N^{1,p}(\Omega)$ and u be a bounded p-harmonic function in Ω. If $E \subset \partial_P\Omega$ such that $\overline{C}_p^P(E) = 0$ and, for all $x \in \partial_P\Omega \setminus E$,*

$$\lim_{\Omega \ni y \xrightarrow{\overline{\Omega}^P} x} u(y) = f(x),$$

then $u = P_{\overline{\Omega}^P} f$.

Proof. Since both f and u are bounded, we may (simultaneously) rescale them such that $0 \le f, u \le 1$. Then we know that $u \in \mathcal{U}_{f-\chi_E}(\overline{\Omega}^P)$ and $u \in \mathcal{L}_{f+\chi_E}(\overline{\Omega}^P)$. Thus, by the preceding proposition,

$$u \le \underline{P}_{\overline{\Omega}^P}(f+\chi_E) = P_{\overline{\Omega}^P} f = \overline{P}_{\overline{\Omega}^P}(f-\chi_E) \le u.$$

□

Finally, as an application of the above resolutivity results, we discuss issues of resolutivity of continuous functions on $\partial_P\Omega$. Note that by the results in [7], continous functions on $\partial\Omega$ are resolutive. However, in the setting of $\partial_P\Omega$ we are unable to get such a general result. Since in general $\partial_P\Omega$ is not metrizable and may not be compact, continuous functions on $\partial_P\Omega$ may not be uniformly approximable by Lipschitz functions on $\partial_P\Omega$. However, we are able to get resolutivity for certain types of continuous functions on $\partial_P\Omega$. This is the focus of the the remaining part of this section.

Recall that $\partial_{SP}\Omega$ denotes the collection of all prime ends whose impression contains only one point. As discussed previously (see also [1]), this set is equipped with a metric d_M, the extension of the Mazurkiewicz metric on Ω.

Proposition 3.3.15. *Let $f\colon \partial_P\Omega \to \mathbb{R}$ be continuous on $\partial_P\Omega$ and d_M-Lipschitz continuous on $\partial_{SP}\Omega$. Then f is resolutive. Furthermore, if $h\colon \partial_P\Omega \to \overline{\mathbb{R}}$ is zero $\overline{C}_p^P$ quasi-everywhere, then $f+h$ is resolutive with respect to $\overline{\Omega}^P$, and $P_{\overline{\Omega}^P}(f+h) = P_{\overline{\Omega}^P}(f)$.*

Proof. By an application of the McShane extension theorem (see [16]), we extend f to a function $F\colon \overline{\Omega}^P \to \mathbb{R}$ such that $F = f$ on $\partial_P\Omega$ and F is d_M-Lipschitz on $\Omega \cup \partial_{SP}\Omega$.

We now show that F is continuous on $\overline{\Omega}^P$. By construction, $F|_{\Omega\cup\partial_{SP}\Omega}$ is continuous. Since $F = f$ on $\partial_P\Omega$, we also see that $F|_{\partial_P\Omega}$ is also continuous. It remains to show that given any end $[\{E_k\}_k] \in \partial_P\Omega \setminus \partial_{SP}\Omega$ and a sequence $\{x_n\}_n$ in Ω with $x_n \overset{\overline{\Omega}^P}{\to} [\{E_k\}_k]$, we have $F(x_n) \to F([\{E_k\}_k])$.

At first, we will prove our result only for sequences $x_n \overset{\overline{\Omega}^P}{\to} [\{E_k\}_k]$ such that, for each n, $x_n \in N(I[\{E_k\}_k], \frac{1}{n})$. In addition, we will fix a representative chain $\{E_k\}_k \in [\{E_k\}_k]$ such that, for all $n \geq k$, $x_n \in E_k$. Recall also that we assume X to be a geodesic space.

By modifying the proof of Theorem 2.3.5, we obtain a sequence $\{[\{F_k^n\}_k]\}_n$ in $\partial_{SP}\Omega$ such that $[\{F_k^n\}_k] \overset{\overline{\Omega}^P}{\to} [\{E_k\}_k]$ and $d_M(x_n, [\{F_k^n\}_k]) \leq \frac{1}{n}$.

Since F is continuous on $\partial_P\Omega$, we know that $F([\{F_k^n\}_k]) \to F([\{E_k\}_k])$. Given any ε, we may pick a large-enough positive integer N such that

$$|F([\{F_k^N\}_k]) - F([\{E_k\}_k])| < \frac{\varepsilon}{2}$$

and

$$|F(x_N) - F([\{F_k^N\}_k])| \leq L d_M(x_N, [\{F_k^N\}_k]) \leq \frac{L}{N} \leq \frac{\varepsilon}{2},$$

where L is the d_M-Lipschitz constant for F on $\Omega \cup \partial_{SP}\Omega$. Then

$$|F(x_N) - F([\{E_k\}_k])| \leq |F([\{F_k^N\}_k]) - F([\{E_k\}_k])| + |F(x_N) - F([\{F_k^N\}_k])| \leq \varepsilon.$$

Thus, $F(x_n) \to F([\{E_k\}_k])$.

Now, given any arbitrary sequence $\{x_n\}$ of points in Ω such that $x_n \overset{\overline{\Omega}^P}{\to} [\{E_k\}_k]$, consider $\{|F(x_n) - F([\{E_k\}_k])|\}_n$. Given any subsequence of $\{x_n\}$, we may pick a further subsequence $\{z_n\}$ such that $z_n \in N(I[\{E_k\}_k], \frac{1}{n})$. Therefore, $|F(z_n) - F([\{E_k\}_k])| \to 0$, implying that $|F(x_n) - F([\{E_k\}_k])| \to 0$, which completes the proof of continuity of F.

Now an application of the main theorem above yields the resolutivity of F, and hence the resolutivity of f, completing the proof of the first part of the proposition.

The second part now follows from an application of Proposition 3.3.13 to the function F. $\square$

Remark 3.3.16. Observe that in the above proposition, we can relax the condition of f being continuous on $\partial_P\Omega$ to f being $\overline{C}_p^P$-quasicontinuous on $\partial_P\Omega$, the remaining (Lipschitz) condition of f also holding. More precisely, if for each $\varepsilon > 0$ we can find an open set $U_\varepsilon \subset \overline{\Omega}^P$ with $\overline{C}_p^P(U_\varepsilon) < \varepsilon$ such that $f|_{[\partial_P\Omega \setminus (U_\varepsilon)] \cup \partial_{SP}\Omega}$ is continuous and f is d_M-Lipschitz continuous on $\partial_{SP}\Omega$, then f is resolutive.

3.4 Some examples

Somewhat surprisingly, the use of prime ends in the Perron method yields new results for Euclidean domains. For example, consider the classical Dirichlet problem, which considers the standard metric boundary. Given continuous boundary data $f : \partial\Omega \to \mathbb{R}$ and a set $E \subset \partial\Omega$ of p-capacity zero, a bounded perturbation of f on E yields a resolutive function whose Perron solution coincides with the Perron solution of f in Ω. Our work with prime ends here allows us to find sets of positive p-capacity which are also exceptional with respect to the Perron solution of f. In this section, we will give 3 examples of such situations in the Euclidean setting.

Example 3.4.1. The first example we discuss in this section is that of the harmonic comb, also known as the topologist's comb. This example was extensively studied in [2]. This comb is the simply connected planar domain given by

$$\Omega := (0,1) \times (0,1) \setminus \bigcup_{n\in\mathbb{N}} \{1/n\} \times [0,1/2].$$

It was shown in [2] that given a function on $\partial\Omega$, continuous and bounded on $\partial\Omega \setminus \{0\} \times [0,1/2)$, any perturbation of the function on the set $E := \{0\} \times [0,1/2)$ yields a resolutive function whose Perron solution coincides with the Perron solution of the original function. Note that $C_p(E) > 0$ for $p > 1$, but $\overline{C}_p^P(P(E)) = 0$. Note also that the prime end boundary in this case is the same as the singleton prime end boundary $\partial_{SP}\Omega$. Hence the "prime end Perron solution" of any boundary data defined on $\partial\Omega$ is independent of the values of the boundary data on E as long as the boundary data is Lipschitz continuous (with respect to the Mazurkiewicz metric d_M) on the part of the boundary of Ω that arises as impressions of prime ends. On the other hand, if f is a quasicontinuous function on $\overline{\Omega} \setminus \{0\} \times [0,1/2)$ (not necessarily bounded) such that $f|_\Omega \in N^{1,p}(\Omega)$, then $f|_{\partial\Omega\setminus\{0\}\times[0,1/2]}$ is resolutive, and any perturbation of f on a set $F \subset \partial\Omega$ with $\overline{C}_p^P(P(F)) = 0$ yields the same Perron solution. Hence the results obtained from the perspective of prime end boundaries are complementary to the results in [2].

In the above example $P(E)$ is empty, as none of the points in E belong to the impression of any prime end. As a consequence of the results of the previous section (see Remark 3.3.16), if we know that $\overline{C}_p^P(\partial_P\Omega \setminus \partial_{SP}\Omega) = 0$, then any bounded function on $\partial_P\Omega$ that is Lipschitz continuous on $\partial_{SP}\Omega$ with respect to the Mazurkiewicz metric d_M is resolutive, and any bounded perturbation of such a function on $\partial_P\Omega \setminus \partial_{SP}\Omega$ yields a resolutive function whose Perron solution agrees with the Perron solution of the original function. This phenomenon is illustrated by the following two examples.

Example 3.4.2. Consider the double harmonic comb defined below:

$$\Omega := (0,1) \times (0,1) \setminus \bigcup_{1<n\in\mathbb{N}} \{1/(2n)\} \times [0, 1-1/n] \cup \{1/(2n+1)\} \times [1/n, 1].$$

This again is a simply connected planar domain, but now the set $E := \{0\} \times [0,1]$ is the impression of a single prime end. Note that $\partial_P\Omega$ is compact in this example, but $\partial_{SP}\Omega$ is not. If we consider functions $u_\varepsilon(\cdot) := \varepsilon \mathrm{dist}^{\Omega}_{inn}(x_0, \cdot)$ for a fixed $x_0 \in \Omega$ and any $\varepsilon > 0$, it is immediate that $\lim\limits_{x \overset{\overline{\Omega}^P}{\to} \partial_P\Omega \setminus \partial_{SP}\Omega} u_\varepsilon(x) \geq 1$, yet $||u_\varepsilon|| \to 0$. Thus $\overline{C}_p^P(\partial_P\Omega \setminus \partial_{SP}\Omega) = 0$, although $C_p(P^{-1}(\partial_P\Omega \setminus \partial_{SP}\Omega)) > 0$. It follows that any function on $\partial_P\Omega$ that is Lipschitz continuous on $\partial_{SP}\Omega$ (with respect to d_M) is resolutive, and any perturbation of this function on E is also resolutive.

We should be careful about notation here, however. Strictly speaking, the entirety of E corresponds to only a single prime end. Thus, by "perturbation on E", we mean to change the function's value to a different one on the entirety of E. We may relax this seemingly strict interpretation as follows. Any function on $\partial\Omega$ that is Lipschitz continuous on $\partial\Omega \setminus E$ is resolutive, and perturbations of such functions on E would yield the same Perron solution.

In the above example we had only one element of the prime end boundary that did not belong to $\partial_{SP}\Omega$. We now construct an example where the set $\partial_P\Omega \setminus \partial_{SP}\Omega$ is uncountable and satisfies $\overline{C}_p^P(\partial_P\Omega \setminus \partial_{SP}\Omega) = 0$ while $C_p(P^{-1}(\partial_P\Omega \setminus \partial_{SP}\Omega)) > 0$.

Example 3.4.3. In this example we consider a domain in $\mathbb{R}^3$:

$$\Omega := (0,1)^3 \setminus \bigcup_{1<n\in\mathbb{N}} \{1/(2n)\}\times[0,3/4+1/n]\times[0,1-1/n] \cup \{1/(2n+1)\}\times[1/4-1/n,1]\times[1/n,1].$$

Clearly none of the points in $E := \{0\} \times [0,1]^2$ is accessible from Ω, and it can be shown using a function similar to the u_ε in the previous example that $\overline{C}_p^P(\partial_P\Omega \setminus \partial_{SP}\Omega) = 0$, while $C_p(E) > 0$. In this case, note that for each non self-intersecting curve γ in E that lies in $\{0\} \times [1/4, 3/4] \times [0,1]$ with endpoints in $\{0\} \times \{1/4\} \times [0,1]$ and $\{0\} \times \{3/4\} \times [0,1]$, there is a prime end in $\partial_P\Omega$ with that curve as its impression. Such a prime end is obtained by considering acceptable sets $E_k = \bigcup_{x\in\gamma} B(x,1/k) \cap \Omega$. By the construction of Ω, it follows that E_k is connected for each positive integer k. It follows that $\partial_P\Omega \setminus \partial_{SP}\Omega$ is uncountable.

CHAPTER 4

Prime Ends of Unbounded Domains

4.1 Prime ends for unbounded domains

We now consider how to define the notion of a prime end of an unbounded domain Ω. Our assumptions on (X, d, μ) remain as they are, we simply now allow Ω to be unbounded. Problems arising from this are twofold. First, proofs in the previous two chapters relied very heavily on Ω being bounded. Second, a decision must be made in how to treat the boundary of Ω "at infinity."

A solution to these problems is found by relating the prime ends of Ω to those of it's image under the sphericalization of X. It is shown in [20] that given a doubling metric measure space (X, d, μ), it's sphericalization $(\dot{X}, \hat{d}, \hat{\mu})$ remains doubling provided X is quasiconvex. Recall that a space is quasiconvex if there is a constant $C \geq 1$ such that for any points $x, y \in X$, there is a curve γ_{xy} connecting x to y such that $\ell(\gamma) \leq Cd(x, y)$.

Given an unbounded domain Ω, we wish to prove a result analogous to Corollary 2.4.4 of Chapter 2. To do so, we wish to define the prime end boundary on Ω such that it is homeomorphic to that of its image under the sphericalization of X. Thus, the purely topological result in Corollary 2.4.4 would transfer immediately. While one could simply define end by way of "pre-images" of the ends sphericalized Ω, an attempt is made in this chapter to create of theory of prime ends for unbounded domains whose definitions are independent of the sphericalization of X.

With this in mind, we now define ends of an unbounded, open and connected set Ω in two parts.

Definition 4.1.1. A connected set $E \subset \Omega$ is said to be *acceptable in* Ω if $\overline{E} \cap \partial\Omega \neq \emptyset$.

Definition 4.1.2. A collection $\{E_k\}_{k\in\mathbb{N}}$ of acceptable subsets of Ω is said to be a *chain in* Ω if

(a) $E_{k+1} \subset E_k$ for all $k \in \mathbb{N}$.

(b) For every k, either $\overline{E_k}$ is compact or there is a compact set $R_k \subset X$, such that $\Omega \setminus R_k$ has two distinct connected components C_k and C_{k+1} with

$$(\partial E_k \setminus R_k) \cap \Omega \subset C_k \text{ and } (\partial E_{k+1} \setminus R_k) \cap \Omega \subset C_{k+1}.$$

(c) For every k, $\operatorname{dist}_M(\partial E_k \cap \Omega, \partial E_{k+1} \cap \Omega) > 0$.

(d) $\emptyset \neq \bigcap_{k=1}^{\infty} \overline{E_k} \subset \partial\Omega$.

Note that if Ω is bounded or E_k is bounded, and X is proper, then Condition (b) is trivially satisfied as long as E_k is not all of Ω. Hence, on bounded domains Ω this definition agrees with that of Chapter 2.

Example 4.1.3. To understand why condition (b) is so vital, consider the domain

$$\Omega = \mathbb{H}^+ \setminus \bigcup_{n=1}^{\infty} [-n, \infty) \times \{\frac{1}{n}\}$$

as a subset of $\mathbb{R}^2$, where $\mathbb{H}^+$ is the upper half plane. If (b) were not a necessary condition for a collection of sets to be a chain, then the collection $\{E_k\}_k$ with $E_k = (-\infty, \infty) \times \{\frac{1}{k}\} \cap \Omega$ would be a chain in Ω. However, there is no chain corresponding to a prime end of Ω which divides $\{E_k\}_k$. Thus, this Ω would fail Assumption 2.3.8.Since this assumption is vital to our use of the Perron method to construct a solution to the Dirichlet problem, we would like to avoid this at all costs.

Figure 4.1: An image of Ω as defined above

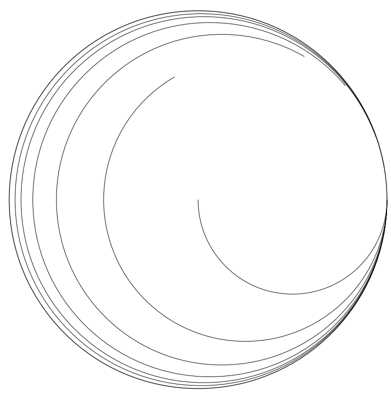

Figure 4.2: An image of the sphericalization of Ω

As an additional motivation, consider the sphericalization of Ω (pictured above). The image of $\{E_k\}_k$ under the canonical injection from Ω to it's sphericalization (see Section 4.2) will not be a valid end. Since we hope do draw an equivalence between the ends of a domain and the ends of its sphericalization, it would be undesirable to consider $\{E_k\}_k$ an end.

We now produce a definition which handles the boundary of Ω "at infinity."

Definition 4.1.4. An unbounded set $E \subset \Omega$ is said to be *acceptable at infinity in* Ω if there exists some compact set $K \subset X$ such that E is a connected component of $\Omega \setminus K$.

Definition 4.1.5. A collection $\{E_k\}_{k\in\mathbb{N}}$ of subsets of Ω which are acceptable at infinity in Ω is said to be a *chain at infinity in* Ω if

(a) $E_{k+1} \subset E_k$ for all $k \in \mathbb{N}$.

(b) For every k, $\mathrm{dist}_M(\partial E_k \cap \Omega, \partial E_{k+1} \cap \Omega) > 0$.

(c) $\bigcap_{k=1}^{\infty} \overline{E_k} = \emptyset$.

Note that specific reference is made to Ω in the objects defined above. This is because in the remainder of this chapter, we will often be considering two separate chains in two separate domains, and so we wish to remember which space we are currently working in.

Remark 4.1.6. From this point on, the term 'chain' will be used to refer to both chains and chains at infinity. In places where only a specific type of chain is considered, this will be explicitly stated.

Definition 4.1.7. Given a chain $\{E_k\}_k$, we define the *Impression of* $\{E_k\}_k$ to be $\bigcap_{k=1}^{\infty} \overline{E_k}$ and denote it $I\{E_k\}_k$. Note that the impression of a chain at infinity is always empty.

Definition 4.1.8. A chain $\{E_k\}_k$ is said to divide another chain $\{F_k\}_k$ if, for every $j \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that $E_n \subset F_j$. The notation $\{E_k\}_k | \{F_k\}_k$ is sometimes used. A similar definition is made for chains at infinity.

Remark 4.1.9. It is immediate that, if $\{E_k\}_k$ and $\{F_k\}_k$ are two chains such that $\{E_k\}_k$ divides $\{F_k\}_k$, then $I\{E_k\}_k \subset I\{F_k\}_k$. Thus we can conclude that a chain at infinity can only be divided by another chain at infinity. An analogous statement for chains not an infinity need not be true, however.

Example 4.1.10. Let $X = \mathbb{R}^2$ and $\Omega = \{(x,y) \in \mathbb{R}^2 \mid y > 0\} \setminus \bigcup_{n=1}^{\infty} S_n$, where $S_n = \Big((-\infty, n] \cup [n+1, \infty)\Big) \times \{2^{-2n}\}$. We may define a chain at infinity $\{E_k\}_k$ as follows. Let $K_k = \Big([k, k+1] \times [2^{-2k-1}, 2^{-2k+1}]\Big) \cup \Big([k, k+\frac{1}{2}] \times [0, 2^{-2k+1}]\Big)$ and E_k be the connected component of $\Omega \setminus K_k$ that contains the point $(n + \frac{3}{2}, 2^{-2n-2})$. If we let

$F_k = \Big(\mathbb{R} \times (0, 2^{-2n})\Big) \cap \Omega$, then $\{F_k\}_k$ is a chain not at infinity (as it fails condition (c) of Definition 4.1.5), yet $\{E_k\}_k$ divides $\{F_k\}_k$.

As in Chapter 2, we may define a natural equivalence relation between chains using divisibility. We say two chains are equivalent if they divide each other. The equivalence class of a chain $\{E_k\}_k$ under this relation is written as $[\{E_k\}_k]$ and is called either an *end* or an *end at infinity*, depending on which type of chains compose the equivalence class. The results of Remark 4.1.9 ensure that this distinction is well-defined.

Remark 4.1.11. In an effort to make ends at infinity easier to work with, we make the following observations. Let $\{E_k\}_k$ be a representative chain of an end at infinity. Thus, each E_k is an unbounded connected component of $\Omega \setminus K_k$, where K_k is some compact set. Fix a point $x_0 \in X$ and let M_k be such that $K_k \subset B(x_0, M_k)$. Since $I\{E_k\}_k = \emptyset$, we may choose an $\ell_k > k$ such that $E_{\ell_k} \subset \Omega \setminus \overline{B(x_0, M_k)}$. Let F_k be the (necessarily unbounded) connected component of $\Omega \setminus \overline{B(x_0, M_k)}$ such that $E_{\ell_k} \subset F_k$. Then, since E_k is a connected component of $\Omega \setminus K_k$, it must be that $F_k \subset E_k$. Therefore, $\{F_k\}_k$ is equivalent to $\{E_k\}_k$.

With a little additional work, we see that given any end at infinity and fixed $x_0 \in X$, said end can be represented by a chain $\{F_k\}_k$, where each F_k is a connected component of $\Omega \setminus \overline{B(x_0, k)}$.

We extend the concept of division to ends and ends at infinity in the natural way. With this, we make the following definitions.

Definition 4.1.12. An end or end at infinity which is only divisible by itself is known as a *prime end* or a *prime end at infinity*, respectively.

Definition 4.1.13. The collection of all ends and ends at infinity of Ω is denoted $\partial_E\Omega$, and is called the *end boundary of* Ω. The collection of all prime ends and prime ends at infinity of Ω is denoted $\partial_P\Omega$, and is called the *prime end boundary of* Ω.

Remark 4.1.14. It should be noted that, due to the observations in Remark 4.1.11, every end at infinity is immediately prime.

We now define a topology on sets $\Omega \cup \partial_E \Omega$ and $\Omega \cup \partial_P \Omega$.

Definition 4.1.15. Given a sequence $\{x_i\}_i$ in Ω and an end $[\{E_k\}_k] \in \partial_E \Omega$, we say that $x_i \overset{\overline{\Omega}^E}{\to} [\{E_k\}_k]$ if for every positive integer k there is a positive integer i_k such that whenever $i \geq i_k$ we have $x_i \in E_k$.

We next extend the topology to $\partial_E \Omega$ by describing sequential topology on $\partial_E \Omega$.

Definition 4.1.16. Given a sequence $\{[\{E_k^n\}_k]\}_n$ of ends of Ω and an end $[\{E_k^\infty\}_k]$ of Ω, we say that $[\{E_k^n\}_k] \overset{\overline{\Omega}^E}{\to} [\{E_k^\infty\}_k]$ if for each positive integer k there is a positive integer n_k such that whenever $n \geq n_k$, there is a positive integer j_n such that $E_{j_n}^n \subset E_k^\infty$.

Definition 4.1.17. Equip the set $\overline{\Omega}^E := \Omega \cup \partial_E \Omega$ with the sequential topology associated with the above notion of limits. Equip the subset $\overline{\Omega}^P := \Omega \cup \partial_P \Omega$ with the subspace topology inherited from $\overline{\Omega}^E$. We call the sets $\overline{\Omega}^E$ and $\overline{\Omega}^P$ the *end closure of* Ω and the *prime end closure of* Ω respectively.

If we wish to discus the closure or boundary of a set $V \subset \Omega$ under the end (or prime end) topology of Ω, we use the notation $\overline{V}^{E,\Omega}$ ($\overline{V}^{P,\Omega}$) and $\partial_E^\Omega V$ ($\partial_P^\Omega V$) respectively.

Remark 4.1.18. Our definition here differs from that given in Chapter 2. It is natural to wonder if the two definitions coincide with respect to bounded domains. It is clear that any chain in Ω is a chain in the sense of Definition 2.2.1. A chain $\{E_k\}_k$ in the sense of Definition 2.2.1 can be seen to immediately satisfy conditions (a), (b) and (d) of Definition 4.1.2, while Remark 2.2.10 discusses the preservation of condition (c), when moving from the regular to Mazurkiewicz distance. Thus, the two definitions are equivalent on bounded domains, and results pertaining to the structure of ends found in Chapter 2 apply in this case.

4.2 Equivalence of the prime ends of a domain and its sphericalization

Let $i\colon X \to \dot{X}$ be the canonical injection of X into its sphericalization, $\dot{X}$. Also let $i^{-1}\colon \dot{X} \setminus \{\infty\} \to X$ be the identity map. Note that both i and i^{-1} are continuous with respect to the metric topologies of X and $\dot{X}$, and that $i(\Omega)$ is a domain in $\dot{X}$. It is also the case that i is locally bi-Lipschitz.

We now lay the foundation to later prove that the spaces $\overline{\Omega}^E$ and $\overline{i(\Omega)}^E$ are homeomorphic. Note that $i(\Omega)$ is a bounded domain, and thus $\partial_E i(\Omega)$ contains no end at infinity.

Lemma 4.2.1. *Let $\{E_k\}_k$ be a chain in Ω. Then $\{i(E_k)\}_k$ is a chain in $i(\Omega)$.*

Proof. Regardless of the structure of $\{E_k\}_k$, it is clear that $i(E_k) \subset i(E_{k+1})$ for each k. If eventually all E_k are bounded, then

$$\bigcap_{k=1}^{\infty} \overline{i(E_k)} = i(\bigcap_{k=1}^{\infty} \overline{E_k}) \subset i(\partial\Omega) \subset \partial i(\Omega).$$

If all E_k are unbounded, then

$$\bigcap_{k=1}^{\infty} \overline{i(E_k)} = i(\bigcap_{k=1}^{\infty} \overline{E_k}) \cup \{\infty\} \subset i(\partial\Omega) \cup \infty = \partial i(\Omega).$$

The above proves that conditions (a) and (d) of Definition 4.1.2 hold for $\{i(E_k)\}_k$. Condition (b) holds immediately, as each $\overline{i(E_k)}$ is compact by the boundedness of $\dot{X}$. To prove that (c) holds, we split into two cases.

Case 1: $\{E_k\}_k$ is a chain at infinity. Therefore, $\partial E_k \cap \Omega$ is bounded for each k (since ∂E_k is a subset of the compact set of which E_k is a connected component of its complement.) Thus, there is a compact set K_k such that $(\partial E_k \cup \partial E_{k+1}) \cap \Omega \subset K_k$.

Since i is bi-Lipschitz within K_k,

$$\text{dist}_{a,M}(\partial i(E_k) \cap i(\Omega), \partial i(E_{k+1}) \cap i(\Omega)) \geq C\text{dist}(\partial E_k \cap \Omega, \partial E_{k+1} \cap \Omega) > 0$$

for some nonzero constant C dependent on k.

Case 2: $\{E_k\}_k$ is not a chain at infinity. If E_k is bounded, then we may proceed similarly to Case 1 above to show that

$$\text{dist}_{a,M}(\partial i(E_k) \cap i(\Omega), \partial i(E_{k+1}) \cap i(\Omega)) > 0.$$

If E_k is unbounded, let R_k, C_k and C_{k+1} be as required in part (b) of Definition 4.1.2. Assume to the contrary that $\text{dist}_{a,M}(\partial i(E_k) \cap i(\Omega), \partial i(E_{k+1}) \cap i(\Omega)) = 0$. Then there must exist sequences $\{x_n\}_n$ and $\{y_n\}_n$ in $i(\Omega)$ such that $\hat{d}_{a,M}(x_n, y_n) < \frac{1}{n}$, with $x_n \in \partial i(E_k) \cap i(\Omega)$ and $y_n \in \partial i(E_{k+1}) \cap i(\Omega)$ for each n. Note that one of the two sequences must converge to $\infty \in \dot{X}$, else the fact that i is locally bi-Lipschitz together with the fact that X is proper would imply that $\text{dist}(\partial E_k \cap \Omega, \partial E_{k+1} \cap \Omega) = 0$. Without loss of generality, we may assume that $x_n \to \infty$. Thus, eventually $x_n \notin i(R_k)$.

Pick an N such that $x_N \notin i(R_k)$ and $\frac{2}{N} < \text{dist}_{a,M}(x_N, i(R_k))$. This implies that $x_N \in i(C_k)$. Thus, since y_N must either be in $\partial i(E_{k+1})$ or $i(R_k)$, any connected set in $i(\Omega)$ which contains both x_N and y_N must intersect $i(R_k)$. Therefore,

$$\frac{1}{N} > \hat{d}_{a,M}(x_N, y_N) \geq \text{dist}_{a,M}(x_N, i(R_k)) > \frac{2}{N},$$

which is impossible. So it must be the case that $\text{dist}_{a,M}(\partial i(E_k) \cap i(\Omega), \partial i(E_{k+1}) \cap i(\Omega)) > 0$. $\square$

Since it is now certain that the image of a chain under i is indeed a chain, we may state the following.

Corollary 4.2.2. *Let $\{E_k\}_k$ and $\{F_k\}_k$ be chains in Ω. Then $\{E_k\}_k$ divides $\{F_k\}_k$ if and only if $\{i(E_k)\}_k$ divides $\{i(F_k)\}_k$.*

The proof of this fact follows clearly from the previous Lemma and the fact that division is a set-theoretic concept clearly preserved by the injection i.

Lemma 4.2.3. *Let $[\{\hat{E}_k\}] \in \partial_E i(\Omega)$. Then there exists a representative chain $\{\hat{F}_k\}_k$ of $[\{\hat{E}_k\}]$ such that $[\{i^{-1}(\hat{F}_k)\}_k] \in \partial_E \Omega$.*

Remark 4.2.4. It should be noted that, due to be Corollary 4.2.2, $[\{i^{-1}(\hat{F}_k)\}_k]$ is unique in the sense that, if $\{\hat{F}'_k\}_k$ is another representative chain of $[\{\hat{E}_k\}]$ with $[\{i^{-1}(\hat{F}'_k)\}_k] \in \partial_E \Omega$, then $[\{i^{-1}(\hat{F}_k)\}_k] = [\{i^{-1}(\hat{F}'_k)\}_k]$.

Proof of Lemma 4.2.3. It suffices to show that there exists a representative chain $\{\hat{F}_k\}_k$ of $[\{\hat{E}_k\}]$ such that $\{i^{-1}(\hat{F}_k)\}_k$ is a chain in Ω.

Since $i(\Omega)$ is a bounded domain, there are no ends at infinity in $i(\Omega)$. However, the end in Ω we wish to associate with $[\{\hat{E}_k\}]$ may potentially be an end at infinity, and thus our proof splits into cases.

Case 1: $\infty \notin I[\{\hat{E}_k\}_k]$.

Since $\infty \notin I[\{\hat{E}_k\}_k]$, there is a representative chain $\{\hat{F}_k\}_k$ of $[\{\hat{E}_k\}_k]$ such that $\infty \notin \overline{\hat{F}_k}$ for any k. Thus, each $i^{-1}(\hat{F}_k)$ is bounded in X. Consider $\{i^{-1}(\hat{F}_k)\}_k$.

Since $I\{\hat{F}_k\} \neq \{\infty\}$, then $\overline{\hat{F}_k} \cap \big(\partial i(\Omega) \setminus \{\infty\}\big) \neq \emptyset$ for every k, and so $\overline{i^{-1}(\hat{F}_k)} \cap \partial\Omega \neq \emptyset$ for every k. Additionally, since i^{-1} is continuous, each $i^{-1}(\hat{F}_k)$ is connected. Thus each $i^{-1}(\hat{F}_k)$ is acceptable.

Thus $i^{-1}(\overline{\hat{F}_k})$ is a compact subset of X with $i^{-1}(\hat{F}_k)$ a connected subset of Ω. Since i is locally biLipschitz, i is biLipschitz in a neighborhood of $i^{-1}(\overline{\hat{F}_1})$, and so we see that the sequence $\{i^{-1}(\hat{F}_k)\}_k$ is a chain in Ω.

Case 2: $\infty \in I[\{\hat{E}_k\}_k]$, but $I[\{\hat{E}_k\}_k] \neq \{\infty\}$.

Let $\{\hat{F}_k\}_k$ be any representative chain of $[\{\hat{E}_k\}_k]$ and consider $\{i^{-1}(\hat{F}_k)\}_k$. The proof that each $i^{-1}(\hat{F}_k)$ is acceptable is identical to the previous case.

Clearly, $\{i^{-1}(\hat{F}_k)\}_k$ is a decreasing sequence fulfilling condition (a) of Definition 4.1.2. Also,

$$\mathrm{dist}_M(\partial i^{-1}(\hat{F}_k) \cap \Omega, \partial i^{-1}(\hat{F}_{k+1}) \cap \Omega) \geq \mathrm{dist}_{a,M}(\partial \hat{F}_k \cap i(\Omega), \partial \hat{F}_{k+1} \cap i(\Omega)) > 0$$

by definition of $\hat{d}_{a,M}$. Additionally, since $I\{\hat{F}_k\} \backslash \{\infty\} \neq \emptyset$, we know that $\bigcap_{k=1}^{\infty} \overline{i^{-1}(\hat{F}_k)}$ is nonempty and must be a subset of $\partial\Omega$. Thus conditions (c) and (d) of Definition 4.1.2 are fulfilled as well.

Since $\infty \in I[\{\hat{E}_k\}_k]$, then $\infty \in \overline{\hat{F}_k}$ for each k. There are now two possibilities regarding the structure of any given $\hat{F}_k$. Either

$$\infty \in \overline{\partial \hat{F}_k \cap i(\Omega)} \tag{4.1}$$

or

$$\infty \notin \overline{\partial \hat{F}_k \cap i(\Omega)} \tag{4.2}$$

Note that if (4.1) holds for k, then it must also hold for $k + 1$. Otherwise, there would be an $\varepsilon_k > 0$ such that

$$B_a(\infty, \varepsilon_k) \cap i(\Omega) \subset \hat{F}_{k+1} \subset \hat{F}_k,$$

which would imply that $\infty \notin \overline{\partial \hat{F}_k \cap i(\Omega)}$, which would be a contradiction.

In the case when (4.1) holds for both k and $k + 1$, there must be a $\tau_k > 0$ such that $\partial \hat{F}_k \cap (B_a(\infty, \tau_k) \cap i(\Omega))$ and $\partial \hat{F}_{k+1} \cap (B_a(\infty, \tau_k) \cap i(\Omega))$ lie in separate connected components of $B_a(\infty, \tau_k) \cap i(\Omega)$, or else $\{\hat{F}_k\}_k$ would fail condition (c) of Definition 4.1.2. Call these connected components T_k and T_{k+1} respectively. Thus, by letting $R_k = i^{-1}\Big(\overline{i(\Omega)} \backslash B_a(\infty, \tau_k)\Big)$, we see that R_k is compact in X and $C_k = i^{-1}(T_k)$

and $C_{k+1} = i^{-1}(T_{k+1})$ are connected components of $\Omega \setminus R_k$ such that

$$(\partial i^{-1}(\hat{F}_k) \setminus R_k) \cap \Omega \subset C_k \text{ and } (\partial i^{-1}(\hat{F}_{k+1}) \setminus R_k) \cap \Omega \subset C_{k+1}.$$

If (4.2) holds for k, we must take more care in constructing R_k. Let δ_k be the minimum of $\mathrm{dist}_{a,M}\Big(\partial \hat{F}_k \cap i(\Omega), \partial \hat{F}_{k+1} \cap i(\Omega)\Big)$ and $\mathrm{dist}_{a,M}\Big(\partial \hat{F}_k \cap i(\Omega), \{\infty\}\Big)$. Note that $\delta_k > 0$. Let

$$T_k = \bigcup_{x \in i(\Omega) \setminus \overline{\hat{F}_k}} B_{a,M}(x, \delta_k/4)$$

and

$$T_{k+1} = \bigcup_{x \in \hat{F}_{k+1}} B_{a,M}(x, \delta_k/4).$$

Note that T_k and T_{k+1} are both connected and necessarily disjoint.

Define R_k as

$$i^{-1}\left(\overline{i(\Omega)} \setminus \Big(T_k \cup T_{k+1} \cup B_a(\infty, \delta_k/4)\Big)\right)$$

and note that R_k is closed and bounded, and thus compact. Note that $\Omega \setminus R_k = i^{-1}\Big(i(\Omega) \setminus i(R_k)\Big)$ and

$$\begin{aligned} i(\Omega) \setminus i(R_k) &= i(\Omega) \setminus \left(\overline{i(\Omega)} \setminus \Big(T_k \cup T_{k+1} \cup B_a(\infty, \delta_k/4)\Big)\right) \\ &= i(\Omega) \setminus \left(i(\Omega) \setminus \Big(T_k \cup T_{k+1} \cup B_a(\infty, \delta_k/4)\Big)\right) \\ &= T_k \cup T_{k+1} \cup (B_a(\infty, \delta_k/4) \cap i(\Omega)). \end{aligned}$$

Immediately, we see that $(\partial \hat{F}_k \setminus i(R_k)) \cap i(\Omega) \subset T_k$ and $(\partial \hat{F}_{k+1} \setminus i(R_k)) \cap i(\Omega) \subset T_{k+1}$. Since both T_{k+1} and $B_a(\infty, \delta_k/4) \cap i(\Omega)$ are disjoint from T_k, it must be the case that T_k is a connected component of $i(\Omega) \setminus i(R_k)$. Therefore, there must exist a connected component T'_{k+1} of $i(\Omega) \setminus R_k$ such that $(\partial \hat{F}_{k+1} \setminus i(R_k)) \cap i(\Omega) \subset T_{k+1} \subset T'_{k+1}$, yet $T'_{k+1} \neq T_k$. Thus we see that $i(\Omega) \setminus i(R_k)$ has two distinct connected components T_k

and T'_{k+1} such that

$$(\partial \hat{F}_k \setminus i(R_k)) \cap i(\Omega) \subset T_k \text{ and } (\partial \hat{F}_{k+1} \setminus i(R_k)) \cap i(\Omega) \subset T'_{k+1}.$$

Thus $\Omega \setminus R_k$ has two distinct connected components $C_k := i^{-1}(T_k)$ and $C_{k+1} := i^{-1}(T'_{k+1})$ such that

$$(\partial i^{-1}(\hat{F}_k) \setminus R_k) \cap \Omega \subset C_k \text{ and } (\partial i^{-1}(\hat{F}_{k+1}) \setminus R_k) \cap \Omega \subset C_{k+1}.$$

Therefore, condition (b) of Definition 4.1.2 is fulfilled for $\{i^{-1}(\hat{F}_k)\}_k$ and $[\{i^{-1}(\hat{F}_k)\}_k] \in \partial_E \Omega$.

Case 3: $I[\{\hat{E}_k\}_k] = \{\infty\}$.

Since $[\{\hat{E}_k\}_k]$ is an end with singleton impression, we know by results of [1] that there exists a representative chain $\{\hat{F}_k\}_k$ of $[\{\hat{E}_k\}_k]$ such that each $\hat{F}_k$ is a connected component of $B_a(\infty, \frac{1}{k+3}) \cap i(\Omega)$ ($\frac{1}{k+3}$ is chosen because the minimum diameter of $\dot{X}$ is $\frac{1}{4}$).

Let $K_k = i^{-1}(\dot{X} \setminus B_a(\infty, \frac{1}{k+1}))$. Then each K_k is compact and, immediately, $i^{-1}(\hat{F}_k)$ is a connected component of $\Omega \setminus K_k$. Thus, each $i^{-1}(\hat{F}_k)$ is acceptable at infinity in Ω.

It is clear that $\{i^{-1}(\hat{F}_k)\}_k$ is a decreasing sequence. Since $\bigcap_{k=1}^{\infty} \hat{F}_k = \{\infty\}$, we see that $\bigcap_{k=1}^{\infty} i^{-1}(\hat{F}_k) = \emptyset$. Finally, note that $(\partial \hat{F}_k \cup \partial \hat{F}_{k+1}) \cap i(\Omega) \subset i(K_{k+1})$. Since $i(K_{k+1})$ is compact and i^{-1} is locally bi-Lipschitz,

$$\mathrm{dist}_M(\Omega \cap \partial i^{-1}(\hat{F}_k), \Omega \cap \partial i^{-1}(\hat{F}_{k+1})) \geq C \mathrm{dist}_{a,M}(i(\Omega) \cap \partial \hat{F}_k, i(\Omega) \cap \partial \hat{F}_{k+1}) > 0$$

for some non-zero constant C which depends on k.

Thus, $\{i^{-1}(\hat{F}_k)\}_k$ is a chain at infinity in Ω and $[\{i^{-1}(\hat{F}_k)\}_k] \in \partial_E \Omega$. $\square$

Theorem 4.2.5. *The spaces $\overline{\Omega}^E$ and $\overline{i(\Omega)}^E$ are homeomorphic.*

Proof. We define two maps $\mathcal{I} : \overline{\Omega}^E \to \overline{i(\Omega)}^E$ and $\mathcal{J} : \overline{i(\Omega)}^E \to \overline{\Omega}^E$ by

$$\mathcal{I}(x) = \begin{cases} i(x), & \text{if } x \in \Omega \\ ; [\{i(E_k)\}_k], & \text{if } x = [\{E_k\}_k] \in \partial_E \Omega; \end{cases}$$

and

$$\mathcal{J}(x) = \begin{cases} i^{-1}(x), & \text{if } x \in i(\Omega) \\ ; [\{i^{-1}(\hat{F}_k)\}_k], & \text{if } x = [\{\hat{E}_k\}_k] \in \partial_E i(\Omega); \end{cases}$$

where $\{\hat{F}_k\}_k$ is the chain representing $[\{\hat{E}_k\}_k]$ whose image under i^{-1} is a chain in Ω. Lemmas 4.2.1 and 4.2.3 and Corollary 4.2.2 give us that $\mathcal{I}$ and $\mathcal{J}$ are well defined and bijective, and Corollary 4.2.2 tells us that that $\mathcal{J}^{-1} = \mathcal{I}$. Basic topology gives us that both $\mathcal{I}$ and $\mathcal{J}$ are continuous with respect to the end topology. Thus, $\overline{\Omega}^E$ and $\overline{i(\Omega)}^E$ are homeomorphic. $\square$

Given this result we may immediately adapt a result from [11].

Corollary 4.2.6. *Let Ω be such that, for every end of Ω there is a prime end of Ω which divides it. Then if $V \subset \Omega$ is a bounded connected, open set, then $\partial V \cap \partial \Omega \neq \emptyset$ if and only if $\partial_P^\Omega V \cap \partial_P \Omega \neq \emptyset$. If V is unbounded, then $\partial_P^\Omega V \cap \partial_P \Omega \neq \emptyset$ always.*

4.3 The Obstacle Problem and p-harmonicity

We examine solutions to the Dirichlet problem on $\overline{\Omega}^P$ by way of the Obstacle Problem. Since our domains under consideration are unbounded, we follow [15] in the use of the Dirichlet space for such an obstacle problem.

Definition 4.3.1. The *Dirichlet space* $D^{1,p}(X)$ is the collection of all measurable functions $f \colon X \to \overline{\mathbb{R}}$ with upper gradients in $L^p(X)$.

However, we will consider a slightly different notion of $D_0^{1,p}(\Omega)$ than does Hansevi.

Definition 4.3.2. Let $D_0^{1,p}(\Omega)$ be the space of all functions $u \in D^{1,p}(\Omega)$ which may be approximated by a sequence of functions $\{u_k\}_k \subset N_C^{1,p}(X) \cap N_0^{1,p}(\Omega)$ in the following way:

(a) $u_k \to u$ in $L^p_{loc}(\Omega)$ and

(b) $\lim_{k\to\infty} \int\limits_\Omega g_{u_k}^p \, d\mu = \int\limits_\Omega g_u^p \, d\mu.$

Note that our definition of $D_0^{1,p}(\Omega)$ is more restrictive than Hansevi's. By demanding that our functions also be the local limit of compactly supported functions in $N_0^{1,p}(\Omega)$, we ask that functions $D_0^{1,p}(\Omega)$ treat infinity as part of the boundary of Ω. Thus, our obstacle problem, which we now define, will enforce a boundary condition at infinity.

Definition 4.3.3. Let $\Omega \subset X$ be an open set with $C_p(X \setminus \Omega) > 0$. Then, for $f \in D^{1,p}(\Omega)$ and $\psi : \Omega \to \overline{\mathbb{R}}$, we define the set

$$\mathcal{K}_{\psi,f}(\Omega) := \{v \in D^{1,p}(\Omega) \mid v - f \in D_0^{1,p}(\Omega),\ v \geq \psi \text{ q.e. in } \Omega\}.$$

A function $u \in \mathcal{K}_{\psi,f}(\Omega)$ is said to be a *solution to the Obstacle Problem in Ω with obstacle ψ and boundary values f* (more succinctly, the solution to the $\mathcal{K}_{\psi,f}(\Omega)$-obstacle problem) if

$$\int\limits_\Omega g_u^p \, d\mu \leq \int\limits_\Omega g_v^p \, d\mu \text{ for all } v \in \mathcal{K}_{\psi,f}(\Omega).$$

Theorem 4.3.4. *Let $\psi : \Omega \to \overline{\mathbb{R}}$ and $f \in D^{1,p}(\Omega)$. Then, if $\mathcal{K}_{\psi,f} \neq \emptyset$, then there exists a unique (up to sets of capacity zero) solution of the $\mathcal{K}_{\psi,f}(V)$ obstacle problem.*

Despite our difference in definition of $D_0^{1,p}(\Omega)$, the proof presented here is nearly identical to that of Hansevi save for a few small details. Instead of simply noting the differences, we display the entire proof here for completeness.

To do so, we must reference the following results from [4, Lemma 3.2] and [4, Corollary 3.3], respectively.

Lemma 4.3.5. *Assume that $1 < p < \infty$. Assume also that $\{u_j\}_{j=1}^{\infty}$ is bounded in $N^{1,p}(X)$ and that $u_j \to u$ quasieverywhere on X as $j \to \infty$. Then $u \in N^{1,p}(X)$ and*

$$\int_X g_u^p \, d\mu \leq \liminf_{j \to \infty} \int_x g_{u_j}^p \, d\mu.$$

Lemma 4.3.6. *Assume that g_j is a p-weak upper gradient of u_j, $j = 1, 2, \dots$, and that both sequences $\{u_j\}_{j=1}^{\infty}$ and $\{g_j\}_{j=1}^{\infty}$ are bounded in $L^p(X)$. Then there are $u, g \in L^p(X)$, convex combinations $v_j = \sum_{i=j}^{N_j} \alpha_{j,i} u_i$ with p-weak upper gradients $\tilde{g}_j = \sum_{i=j}^{N_j} \alpha_{j,i} g_i$ and a subsequence $\{u_{j_k}\}_{k=1}^{\infty}$, such that*

(a) both $u_{j_k} \to u$ and $g_{j_k} \to g$ weakly in $L^p(X)$,

(b) both $v_j \to u$ and $\tilde{g}_j \to g$ in $L^p(X)$,

(c) $v_j \to u$ q.e.

(d) g is a p-weak upper gradient of u.

Proof of Theorem 4.3.4. Let

$$I = \inf_{v \in \mathcal{K}_{\psi,f}(\Omega)} \int_\Omega g_v^p \, d\mu.$$

Since $\mathcal{K}_{\psi,f}(\Omega)$ is nonempty, we know that $0 \leq I < \infty$. Let $\{u_j\}_j \subset \mathcal{K}_{\psi,f}(\Omega)$ be a minimizing sequence such that

$$I_j := \int_\Omega g_{u_j}^p \, d\mu \searrow I \text{ a } j \to \infty.$$

In particular, since $u_j - f \in D_0^{1,p}(\Omega)$, we may choose these u_j such that $w_j := u_j - f \in N_C^{1,p}(X) \cap N_0^{1,p}(\Omega)$. Also note that g_{u_j} is necessarily bounded in $L^p(\Omega)$.

Since

$$||g_{w_j}||_{L^p(X)} \leq ||g_{u_j}||_{L^p(\Omega)} + ||g_f||_{L^p(\Omega)}$$

and g_{u_j} is bounded in $L^p(\Omega)$, we see that g_{w_j} is bounded in $L^p(X)$.

Let B be a ball such that $\mu(B \setminus \Omega) > 0$. For $n \in \mathbb{N}$, by nB we denote the ball concentric to B with radius $n\mathrm{rad}(B)$. Note that $\mu(nB \setminus \Omega) > 0$ as well. Thus, by Maz'ya's inequality, for each $n \in \mathbb{N}$ there is a constant $C_n > 0$ such that

$$\int\limits_{nB} |w_j|^p \, d\mu \leq C_n^p \int\limits_{\lambda nB} g_{w_j}^p \, d\mu.$$

Thus

$$||w_j||_{L^p(nB)} \leq C_n ||g_{w_j}||_{L^p(\lambda nB)} \leq ||g_{w_j}||_{L^p(X)},$$

and therefore, for each $n \in \mathbb{N}$, w_j is bounded in $L^p(nB)$.

By lemma 4.3.6, we may find a function $\phi_1 \in L^p(B)$ and a convex combination sequence

$$\phi_{1,j} := \sum_{k=j}^{N_{1,j}} \alpha_{1,j,k} w_k \text{ in } D^p(X),$$

such that $\phi_{1,j} \to \phi_1$ q.e. in B as $j \to \infty$.

Let $v_{1,j} = f + \phi_{1,j}|_\Omega$. Then

$$v_{1,j} = f + \sum_{k=j}^{N_{1,j}} \alpha_{1,j,k} w_k|_\Omega = \sum_{k=j}^{N_{1,j}} \alpha_{1,j,k} (f + w_k)|_\Omega = \sum_{k=j}^{N_{1,j}} \alpha_{1,j,k} u_k \geq \psi \text{ q.e. in } \Omega.$$

Additionally,

$$g_{v_{1,j}} \leq \sum_{k=j}^{N_{1,j}} \alpha_{1,j,k} g_{u_k} \text{ a.e. in } \Omega \text{ and } g_{\phi_{1,j}} \leq \sum_{k=j}^{N_{1,j}} \alpha_{1,j,k} g_{w_k} \text{ a.e.}$$

Note that $\{\phi_{1,j}\} \subset N_C^{1,p}(X) \cap N_0^{1,p}(\Omega)$. Additionally, both $\{\phi_{1,j}\}$ and $\{g_{\phi_{1,j}}\}$ remain bounded sequences in $L^p(nB)$ for each $n \in \mathbb{N}$. Thus, we may repeat the above process to create further convex combination subsequences of $\{\phi_{1,j}\}$ and $\{g_{\phi_{1,j}}\}$. In fact, for every $n \geq 1$ we may find a function $\phi_n \in L^p(nB)$ and a convex combination sequence

$$\phi_{n,j} := \sum_{k=j}^{N_{n,j}} \alpha_{n,j,k}\phi_{n-1,k} \text{ in } D^p(X),$$

such that $\phi_{n,j} \to \phi_n$ q.e. in nB as $j \to \infty$.

Let $v_{n,j} = f + \phi_{n,j}|_\Omega$. Then

$$v_{n,j} = \sum_{k=j}^{N_{n,j}} \alpha_{n,j,k}(f + \phi_{n-1,k})|_\Omega = \sum_{k=j}^{N_{n,j}} \alpha_{n,j,k} n_{n-1,k} \geq \psi \text{ q.e. in } \Omega.$$

Additionally,

$$g_{v_{n,j}} \leq \sum_{k=j}^{N_{n,j}} \alpha_{n,j,k} g_{v_{n-1,k}} \text{ a.e. in } \Omega \text{ and } g_{\phi_{n,j}} \leq \sum_{k=j}^{N_{n,j}} \alpha_{n,j,k} g_{\phi_{n-1,k}} \text{ a.e.}$$

Define ϕ as a function on X by

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n(x)\chi_{nB\setminus(n-1)B}(x), \; x \in X,$$

and let $u = f + \phi|_\Omega$. Our goal now is to show that this u is our solution to the $\mathcal{K}_{\phi,f}$ obstacle problem. To do so, we must show that $\phi \in D_0^{1,p}(\Omega)$.

Consider the sequences $\{v_{n,n}\}$ and $\{\phi_{n,n}\}$. For any fixed $n \geq 1$,

$$\begin{aligned} |\phi_{n+1} - \phi_n| &\leq |\phi_{n+1} - \phi_{n+1,j}| + |\phi_{n+1,j} - \phi_n| \\ &\leq |\phi_{n+1} - \phi_{n+1,j}| + \sum_{k=j}^{N_{n+1,j}} \alpha_{n+1,j,k}|\phi_{n,k} - \phi_n| \to 0 \end{aligned}$$

q.e. in nB as $j \to \infty$. Therefore, $\phi_{n+1} = \phi_n$ in nB for all $n \in \mathbb{N}$. A simple inductive argument shows that $\phi = \phi_n$ q.e. in nB for each $n \in \mathbb{N}$.

We now show that $\phi_{n,n} \to \phi$ q.e., and thus that $v_{n,n} \to u$ q.e. in Ω as well.

Let E_n be the subset of nB where $\phi_{n,j} \to \phi$ as $j \to \infty$ and let $E = \bigcup_{n=1}^{\infty}(nB \setminus E_n)$. Then $C_p(E) \leq \sum_{n=1}^{\infty} C_p(nB \setminus E_n) = 0$. Consider $x \in \setminus E$. Then for, some $m \in \mathbb{N}$, $x \in mB$ and thus $\phi(x) = \phi_m(x)$. Given $\varepsilon > 0$, let J be such that for all $j \geq J$,

$$|\phi_{m,j}(x) - \phi_m(x)| < \varepsilon.$$

If, for some $n \geq m$, we have that $|\phi_{n,j}(x) - \phi_m(x)| < \varepsilon$ for $j \geq J$, then

$$|\phi_{n+1,j}(x) - \phi_m(x)| \leq \sum_{k=j}^{N_{n+1,j}} \alpha_{n+1,j,k} |\phi_{n,k}(x) - \phi_m(x)| < \varepsilon$$

for $j \geq J$. Thus, inductively, we see that $|\phi_n, j(x) - \phi_m(x)| < \varepsilon$ for $n \geq m$ and $j \geq J$, and thus, for $n \geq \max\{m, J\}$, we have that

$$|\phi_{n,n}(x) - \phi(x)| = |\phi_{n,n}(x) - \phi_m(x)| < \varepsilon.$$

Thus $\phi_{n,n} \to \phi$ q.e., and so $v_{n,n} \to u$ q.e. in Ω as well.

By Jensen's Inequality, we see that

$$\int\limits_{\Omega} g_{v_{1,j}}^p \, d\mu \leq \int\limits_{\Omega} \left(\sum_{k=j}^{N_{1,j}} \alpha_{1,j,k} g_{u_k} \right)^p \, d\mu \leq \sum_{k=j}^{N_{1,j}} \alpha_{1,j,k} \int\limits_{\Omega} g_{u_k} \, d\mu \leq \int\limits_{\Omega} g_{u_j}^p \, d\mu$$

and

$$\begin{aligned} \int\limits_{X} g_{\phi_{1,j}}^p \, d\mu &\leq \int\limits_{X} \left(\sum_{k=j}^{N_{1,j}} \alpha_{1,j,k} g_{w_k} \right)^p \, d\mu \leq \sum_{k=j}^{N_{1,j}} \alpha_{1,j,k} \int\limits_{\Omega} (g_{u_k} + g_f)^p \, d\mu \\ &\leq 2^p \sum_{k=j}^{N_{1,j}} \alpha_{1,j,k} \int\limits_{\Omega} (g_{u_k}^p + g_f^p) \, d\mu \leq 2^p \int\limits_{\Omega} (g_{u_k}^p + g_f^p) \, d\mu. \end{aligned}$$

If, for some $n \in \mathbb{N}$, we know that

$$\int\limits_{\Omega} g_{v_{n,j}}^p \, d\mu \le \int\limits_{\Omega} g_{u_j}^p \, d\mu \text{ and } \int\limits_{X} g_{\phi_{n,j}}^p \, d\mu \le 2^p \int\limits_{\Omega} (g_f^p + g_{u_j}^p) \, d\mu,$$

then we also know that

$$\begin{aligned}
\int\limits_{\Omega} g_{v_{n+1,j}}^p \, d\mu &\le \int\limits_{\Omega} \left(\sum_{k=j}^{N_{n+1,j}} \alpha_{n+1,j,k} g_{v_{n,k}} \right)^p d\mu \le \sum_{k=j}^{N_{n+1,j}} \alpha_{n+1,j,k} \int\limits_{\Omega} g_{v_{n,k}}^p \, d\mu \\
&\le \sum_{k=j}^{N_{n+1,j}} \alpha_{n+1,j,k} \int\limits_{\Omega} g_{u_k}^p \, d\mu \le \int\limits_{\Omega} g_{u_j}^p \, d\mu
\end{aligned}$$

and

$$\begin{aligned}
\int\limits_{X} g_{\phi_{n+1,j}}^p \, d\mu &\le \int\limits_{X} \left(\sum_{k=j}^{N_{n+1,j}} \alpha_{n+1,j,k} g_{\phi_{n,k}} \right)^p d\mu \le \sum_{k=j}^{N_{n+1,j}} \alpha_{n+1,j,k} \int\limits_{\Omega} g_{\phi_{n,k}}^p \, d\mu \\
&\le 2^p \sum_{k=j}^{N_{1,j}} \alpha_{1,j,k} \int\limits_{\Omega} g_{u_k}^p + g_f^p \, d\mu \le 2^p \int\limits_{\Omega} (g_{u_k}^p + g_f^p) \, d\mu.
\end{aligned}$$

Thus, by induction, we know that, for all $n \in \mathbb{N}$,

$$\int\limits_{\Omega} g_{v_{n,n}}^p \, d\mu \le \int\limits_{\Omega} g_{u_n}^p \, d\mu \text{ and } \int\limits_{X} g_{\phi_{n,n}}^p \, d\mu \le 2^p \int\limits_{\Omega} (g_f^p + g_{u_n}^p) \, d\mu.$$

For any fixed $m \in \mathbb{N}$, we know that $\{\phi_{n,n}\}$ and $\{g_{\phi_{n,n,}}\}$ are both bounded in $L^p(mB)$ and $\phi_{n,n} \to \phi$ q.e. in mB, thus $\phi \in N^{1,p}(mB)$. Thus, $\phi \in D^{1,p}_{loc}(\Omega)$ with minimal p-weak upper gradients ϕ and $\phi_{n,n}$ respectively in mB. Thus,

$$\begin{aligned}
\int\limits_{mB} g_\phi^p \, d\mu &\le \liminf_{n\to\infty} \int\limits_{mB} g_{\phi_{n,n}}^p \, d\mu \le \liminf_{n\to\infty} \int\limits_{X} g_{\phi_{n,n}}^p \, d\mu \\
&\le 2^p \liminf_{n\to\infty} \int\limits_{\Omega} (g_f^p + g_{u_n}^p) \, d\mu = 2^p \int\limits_{\Omega} g_f^p \, d\mu + 2^p I.
\end{aligned}$$

Thus, by letting $m \to \infty$, we see that

$$\int\limits_X g_\phi^p \, d\mu = \lim_{m\to\infty} \int\limits_{mB} g_\phi^p \, d\mu \le 2^p \int\limits_\Omega g_f^p + 2^p I < \infty.$$

Thus, $\phi \in D^{1,p}(X)$.

Let

$$\Omega_t = \left\{ x \in tB \cap \Omega \mid \inf_{x\in\partial\Omega} d(x,y) > \frac{\delta}{t} \right\}, \ 1 \le t < \infty,$$

with $\delta > 0$ chosen such that Ω_1 is nonempty. Then $\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset \Omega = \bigcup_{t=1}^\infty \Omega_t$.

Since $f \in D^{1,p}(\Omega)$, we know that $f \in N^{1,p}_{loc}(\Omega)$. Thus, $f \in L^p(\Omega_t)$ for every t. Given any $m \in \mathbb{N}$, since both $\{v_{n,n}\}$ and $\{g_{v_{n,n}}\}$ are bounded in $L^p(\Omega_m)$ and $v_{n,n}$ q.e. in Ω_m, we know that

$$\int\limits_{\Omega_m} g_u^p \, d\mu \le \liminf_{n\to\infty} \int\limits_{\Omega_m} g_{v_{n,n}}^p \, d\mu \le \liminf_{n\to\infty} \int\limits_\Omega g_{v_{n,n}}^p \, d\mu \le \liminf_{n\to\infty} \int\limits_\Omega g_{u_n}^p \, d\mu = I.$$

By letting $m \to \infty$, we see that

$$I \le \int\limits_\Omega g_u^p \, d\mu = \lim_{m\to\infty} \int\limits_{\Omega_m} g_u^p \, d\mu \le I.$$

Thus u attains minimal energy.

This also shows that $\lim_{n\to\infty} \int_\Omega g_{\phi_{n,n}}^p \, d\mu = \int_\Omega g_\phi \, d\mu$, and thus $\phi \in D_0^{1,p}(\Omega)$.

Finally, let $A_n = \{x \in \Omega \mid v_{n,n}(x) < \phi(x)\}$ and let $A = \bigcup_{n=1}^\infty A_n$. Then, since $v_{n,n} \to v$ q.e. in Ω, then $u \ge \phi$ in $\Omega \setminus A$. Since $v_{n,n} \ge \phi$ q.e. in Ω, it must be that $C_p(A_n) = 0$ and thus $C_p(A) = 0$. So $u \ge \phi$ q.e. in Ω. Thus, $u \in \mathcal{K}_{\psi,f}$ and so u is a solution to the $\mathcal{K}_{\psi,f}$ obstacle problem.

We now show that this solution is unique.

Let u' and u'' be solutions to the $\mathcal{K}_{\psi,f}$ obstacle problem. Then $\frac{1}{2}(u' + u'') \in \mathcal{K}_{\psi,f}$, thus

$$\begin{aligned}||g_{u'}||_{L^p(\Omega)} &\le ||g_{\frac{1}{2}(u'+u'')}||_{L^p(\Omega)} \le ||\frac{1}{2}(g_{u'}+g_{u''})||_{L^p(\Omega)} \\ &\le \frac{1}{2}||g_{u'}||_{L^p(\Omega)} + \frac{1}{2}||g_{u''}||_{L^p(\Omega)} = ||g_{u'}||_{L^p(\Omega)} = ||g_{u''}||_{L^p(\Omega)}.\end{aligned}$$

So, by the strict convexity of $L^p(\Omega)$, we know that $g_{u'} = g_{u''}$ a.e. in Ω. Fix a $c \in \mathbb{R}$ and let

$$u = \max\{u', \min\{u'', c\}\}.$$

Immediately, $u \in D^{1,p}(\Omega)$ and $u \ge u' \ge \phi$ q.e. in Ω. Since

$$u - f \le \max\{u', u''\} - f = \max\{u' - f, u'' - f\} \in D_0^{1,p}(\Omega)$$

and $u - f \ge u' - f \in D_0^{1,p}(\Omega)$, then $u - f \in D_0^{1,p}(\Omega)$.

Let $U_c = \{x \in \Omega \mid u'(x) < c < u''(x)\}$. Then $g_u = 0$ a.e. in U_c, since $U_c \subset \{x \in \Omega \mid u(x) = c\}$. Since $g_{u'}$ is a minimizer and $g_u = g_{u'} = g_{u''}$ a.e. in $\Omega \setminus U_c$,

$$\int_{\Omega} g_{u'}^p \, d\mu \le \int_{\Omega} g_u^p \, d\mu = \int_{\Omega\setminus U_c} g_u^p \, d\mu = \int_{\Omega\setminus U_c} g_{u'}^p \, d\mu.$$

So $g_{u'} = g_{u''} = 0$ a.e. in U_c for all $c \in \mathbb{R}$. Since

$$\{x \in \Omega \mid u'(x) < u''(x)\} \subset \bigcup_{c\in\mathbb{Q}} U_c,$$

we know that $g_{u'} = g_{u''} = 0$ a.e. in $\{x \in \Omega \mid u'(x) < u''(x)\}$. Since the labeling of u' and u'' is arbitrary, we also know that $g_{u'} = g_{u''} = 0$ a.e. in $\{x \in \Omega \mid u'(x) > u''(x)\}$. Thus,

$$g_{u'-u''} \le (g_{u'} + g_{u''})\chi_{\{x\in\Omega \mid u'(x)\neq u''(x)\}} = 0 \text{ a.e. in } \Omega.$$

Thus, by Maz'ya's inequality, for each $n \in \mathbb{N}$, there is a C_n such that

$$\int_{nB\cap\Omega} |u' - u''|^p \, d\mu \le C_n^p \int_{\lambda nB} g_{u'-u''}^p \, d\mu = 0.$$

Therefore, $u' = u''$ q.e. in $nB \cap \Omega$. Thus, $u' = u''$ q.e. in Ω, and uniqueness is shown. □

We will primarily be concerned with the case when $\psi \equiv -\infty$. In this case the obstacle problem simply becomes an issue of minimizing energy in Ω while attaining the given boundary values f, transforming the problem into the Dirichlet problem. In this case, $\mathcal{K}_{-\infty,f}$ is clearly non-empty, and so we denote the solution to the $\mathcal{K}_{-\infty,f}(\Omega)$-obstacle problem by $H_\Omega f$ and, when no ambiguity would arise, we will refer to it simply as Hf. We will also refer to the solution of the $\mathcal{K}_{f,f}$ obstacle problem, which is guaranteed to exists by the trivial nonemptiness of $\mathcal{K}_{f,f}$.

We also introduce here a new capacity for sets in $\overline{\Omega}^P$. This capacity is based on the capacity introduced and discussed extensively in Chapter 2, so we will cite results from there as needed.

First we partition $\partial_P\Omega$ into two distinct parts.

Definition 4.3.7. Denote

$$\partial_P^F\Omega := \{x \in \partial_P\Omega \mid I[x] \text{ is bounded and nonempty}\}$$

as the *finite prime end boundary.* Denote

$$\partial_P^\infty\Omega := \partial_P\Omega \setminus \partial_P^F\Omega$$

as the *infinite prime end boundary.*

Note that, despite its name, $\partial_P^\infty\Omega$ does not necessarily contain only ends at infinity.

Definition 4.3.8. Let $E, F \subset \overline{\Omega}^P$ with $E \subset \overline{\Omega}^P \setminus F$. Then

$$\overline{C_p^P}(E : F; \Omega) := \inf_{u \in \mathcal{A}_{E,F}} ||u||^p_{D^{1,p}(\Omega)}$$

where $u \in \mathcal{A}_{E,F}$ if $u = 0$ in $E \cap \Omega$,

$$\limsup_{\Omega \ni y \overset{\overline{\Omega}^P}{\to} x} u(y) \leq 0 \text{ for all } x \in E \cap \partial_P \Omega,$$

$u = 1$ in F,

$$\liminf_{\Omega \ni y \overset{\overline{\Omega}^P}{\to} x} u(y) \geq 1 \text{ for all } x \in F \cap \partial_P \Omega,$$

and $0 \leq u \leq 1$ in Ω.

Definition 4.3.9. Let $E \subset \overline{\Omega}^P$. Fix $a \in \Omega$ and $r > 0$ such that $\overline{B(a,r)} \subset \Omega$. Then

$$\overline{C_p^{P,\infty}}(E) := \lim_{n \to \infty} \overline{C_p^P}(B(a,r) : E \setminus P(B(a,nr)); \Omega)$$

Proposition 4.3.10. *Suppose that the measure on X is doubling and supports a p-Poincaré inequality. Then $\overline{C_p^{P,\infty}}$ is an outer capacity, i.e. for all $E \subset \overline{\Omega}^P$,*

$$\overline{C_p^{P,\infty}}(E) = \inf_G \overline{C_p^{P,\infty}}(G),$$

where the infimum is taken over all $G \supset E$ that are open in $\overline{\Omega}^P$.

Proof. We know that $\overline{C}_p^P(B(a,r), E \setminus P(B(a,nr)); \Omega)$ is monotone with respect to E, and thus $\overline{C}_p^P(B(a,r), E \setminus P(B(a,nr)); \Omega) \leq \inf_G \overline{C}_p^P(B(a,r), G \setminus P(B(a,nr)); \Omega)$.

Fix an n. Given $E \subset \overline{\Omega}^P$ and $\varepsilon > 0$, we pick a function $u \in \mathcal{A}_{B(a,r),E \setminus P(B(a,nr))}$ with $0 \leq u \leq 1$ such that

$$||u||_{D^{1,p}(\Omega)} \leq \overline{C}_p^P(B(a,r), E \setminus P(B(a,nr)); \Omega)^{1/p} + \varepsilon.$$

Since u is quasicontinuous on Ω, we may also take some open set $V \subset \Omega$ such that $C_p(V)^{1/p} \leq \varepsilon$ and $u|_{\Omega\setminus V}$ is continuous. Thus, $\{x \in \Omega | u(x) > 1-\varepsilon\} \setminus V$ is an open set in $\Omega \setminus V$ with respect to the subspace topology. Therefore there is another open set $U \subset \Omega$ such that

$$U \setminus V = \{x \in \Omega | u(x) > 1-\varepsilon\} \setminus V \supset ((E \cap \Omega) \setminus B(a, nr)) \setminus V.$$

Because $C_p(V) \leq \varepsilon^p$, we can choose $v \in N^{1,p}(X)$ satisfying $||v||_{D^{1,p}(X)} \leq ||v||_{N^{1,p}(X)} < 2\varepsilon$, $0 \leq v \leq 1$ on X, and $v \geq 1$ on V. Set

$$w = \frac{u}{1-\varepsilon} + v.$$

Then $w \geq 1$ on $U \cup V$, which is an open set containing $(E \cap \Omega) \setminus B(a, nr)$. Also, for each $[\{E_k\}_k] \in (E \setminus P(B(a, nr))) \cap \partial_P \Omega$, there is a positive integer K such that $u > 1-\varepsilon$ on E_K. Indeed, if not, then we can find a sequence of points $x_k \in E_k$ such that $u(x_k) \leq 1-\varepsilon$ but $x_k \overset{\overline{\Omega}^P}{\to} [\{E_k\}_k] \in (E \setminus P(B(a, nr))) \cap \partial_P \Omega$, a violation of the choice of $u \in \mathcal{A}_{B(a,r),E\setminus P(B(a,nr))}$. Note that $w \geq 1$ on E_K.

Let

$$W = U \cup V \cup \bigcup_{[\{E_k\}_k] \in E \cap \partial_P \Omega} (E_K \cup E_K^P),$$

where E_K^P is as defined in Remark 2.2.12. Here, we have chosen the E_ks to be open. Then $W \supset (E \setminus P(B(a, nr)))$ is an open set in $\overline{\Omega}^P$ and $w \in \mathcal{A}_{B(a,r),W\setminus P(B(a,nr))}$. So

$$\begin{aligned}
\overline{C}_p^P(B(a,r), E\setminus P(B(a,nr)); \Omega)^{1/p} &\leq \inf_G \overline{C}_p^P(B(a,r), G \setminus P(B(a,nr)); \Omega)^{1/p} \\
&\leq \overline{C}_p^P(B(a,r), W \setminus P(B(a,nr)); \Omega)^{1/p} \leq ||w||_{N^{1,p}(\Omega^P)} \\
&\leq \frac{1}{1-\varepsilon} ||u||_{D^{1,p}(\Omega^P)} + ||v||_{D^{1,p}(\Omega^P)} \\
&\leq \frac{1}{1-\varepsilon} (\overline{C}_p^P(B(a,r), E \setminus P(B(a,nr)); \Omega)^{1/p} + \varepsilon) + 2\varepsilon.
\end{aligned}$$

By letting $\varepsilon \to 0$, we see that

$$\overline{C}_p^P(B(a,r), E \setminus P(B(a,nr)); \Omega) = \inf_G \overline{C}_p^P(B(a,r), G \setminus P(B(a,nr)); \Omega)$$

for every $n \in \mathbb{N}$, and thus $\overline{C_p^{P,\infty}}(E) = \inf_G \overline{C_p^{P,\infty}}(G)$, as desired. $\square$

We now combine the above definitions to create a new capacity on $\overline{\Omega}^P$.

Definition 4.3.11. For $E \subset \overline{\Omega}^P$, let

$$\overline{C_p^P}(E) = \max\{\overline{C}_p^P(E), \overline{C_p^{P,\infty}}(E)\}.$$

Since both $\overline{C}_p^P$ and $\overline{C_p^{P,\infty}}$ are outer capacities, we immediately know that $\overline{C_p^P}$ is also an outer capacity. Note that when Ω is bounded, $\overline{C_p^P} \equiv \overline{C}_p^P$.

Note that, when $E \subset X$ is bounded, $\overline{C_p^{P,\infty}}(E) = 0$, and so we know that $\overline{C_p^P}(P(E)) \leq C_p(E)$ as before.

We now define what it means for a function to be $\overline{C_p^P}$-quasicontinuous.

Definition 4.3.12. A function $f\colon \overline{\Omega}^P \to \overline{\mathbb{R}}$ is said to be $\overline{C_p^P}$-*quasicontinuous* if, for every $\varepsilon > 0$, there is a relatively open set $U \subset \overline{\Omega}^P$ such that $\overline{C_p^P}(U) \leq \varepsilon$ and $f|_{\overline{\Omega}^P \setminus U}$ is real-valued continuous.

4.4 The Perron solution with respect to prime ends in unbounded domains

For convenience, we restate the definitions of superharmonicity and the Perron solution here.

Definition 4.4.1. A function $u\colon \Omega \to (-\infty, \infty]$ is said to be p-superharmonic if

(a) u is lower semicontinuous,

(b) u is not identically ∞ on Ω, and

(c) for every nonempty open set $V \subset\subset \Omega$ and all functions $f \in Lip(X)$, if $v \leq u$ on ∂V, then $H_V v \leq u$ in V.

A function u is said to be p-subharminic if $-u$ is p-superharmonic.

Definition 4.4.2. Given a function $f\colon \partial_P\Omega \to \overline{\mathbb{R}}$, let $\mathcal{U}_f(\overline{\Omega}^P)$ be the set of all p-superharmonic functions u on Ω bounded below such that

$$\liminf_{\Omega \ni \overset{\overline{\Omega}^P}{\to} x} u(y) \geq f(x) \text{ for all } x \in \partial_P\Omega.$$

We define the *upper Perron solution* of f by

$$\overline{P}_{\overline{\Omega}^P} f(x) := \inf_{u \in \mathcal{U}_f(\overline{\Omega}^P)} u(x),\ x \in \Omega.$$

Similarly, let $\mathcal{L}_f(\overline{\Omega}^P)$ be the set of all p-subharmonic functions u on Ω bounded above such that

$$\limsup_{\Omega \ni \overset{\overline{\Omega}^P}{\to} x} u(y) \leq f(x) \text{ for all } x \in \partial_P\Omega.$$

We define the *lower Perron solution* of f by

$$\underline{P}_{\overline{\Omega}^P} f(x) := \sup_{u \in \mathcal{L}_f(\overline{\Omega}^P)} u(x),\ x \in \Omega.$$

Note that $\underline{P}_{\overline{\Omega}^P} f = \overline{P}_{\overline{\Omega}^P}(-f)$. If $\overline{P}_{\overline{\Omega}^P} f = \underline{P}_{\overline{\Omega}^P} f$ in Ω, then we let $P_{\overline{\Omega}^P} f := \overline{P}_{\overline{\Omega}^P} f$, call $P_{\overline{\Omega}^P} f$ the *Perron solution* of f in Ω, and f is said to be *resolutive.*

We now make the following assumption. This will allow us to import directly our results about the structure of the prime end boundary made in Chapter 2. Namely, it will allow the use of Corollary 2.4.4. For convenience, we will restate this Corollary after the assumption.

Assumption 4.4.3. We assume that every end (at infinity or otherwise) of Ω has a prime end of Ω which divides it.

Corollary 4.4.4. *If $V \subset \Omega$ is a bounded open and connected set, then $\overline{V} \cap \partial_\Omega \neq \emptyset$ if and only if $\overline{V}^{\Omega,P} \cap \partial_P\Omega \neq \emptyset$.*

Theorem 4.4.5. *Suppose that u is superharmonic and that v is subharmonic in Ω. If*

$$\limsup_{y \overset{\overline{\Omega}^P}{\to} x} v(y) \leq \liminf_{y \overset{\overline{\Omega}^P}{\to} x} u(y)$$

for every $x \in \partial_P\Omega$ and both sides are not simultaneously infinite, then $v \leq u$ in Ω.

Proof. Let $\varepsilon > 0$. For each $x \in \partial_P\Omega$, let E_x be an open, acceptable set which is a member of a chain representing x and $v \leq u + \frac{\varepsilon}{2}$ in E_x. Then $\Omega_\varepsilon := \Omega \setminus \left(\overline{\bigcup_{x \in \partial_P\Omega} E_x} \right)$ is an open set such that $v < u + \varepsilon$ on $\partial\Omega_\varepsilon$. Note that Ω_ε may be unbounded, however we will show that, by the above corollary, each connected component of Ω_ε is bounded.

Assume there is is an unbounded connected component V of Ω_ε, then consider $i(V) \subset i(\Omega)$. Since V is unbounded and $\overline{V} \cap \Omega = \emptyset$, then $i(V) \cap \partial_P i(\Omega) = \{\infty\}$. By the above corollary, $\overline{V}^{i(\Omega),P} \cap \partial_P i(\Omega)$ cannot be empty. Thus, by the equivalence of the prime ends in Ω and in $i(\Omega)$, $\overline{V}^{\Omega,P} \cap \partial_P\Omega$ cannot be empty either. Thus $\overline{\Omega_\varepsilon}^{\Omega,P} \cap \partial_P\Omega \neq \emptyset$, which contradicts the construction of Ω_ε. Thus, every connected component of Ω_ε must be bounded. Therefore, we may proceed as in Theorem 7.2 of [18] for each connected component of Ω_ε, showing that $v \leq u$ in Ω_ε. By exhausting Ω with such sets, we see that $v \leq u$ in all of Ω. $\square$

Corollary 4.4.6. *If $f : \partial_P\Omega \to \mathbb{R}$, then*

$$\underline{P}_{\overline{\Omega}^P} f \leq \overline{P}_{\overline{\Omega}^P} f.$$

We restate Lemma 3.3.7 for convenience here.

Lemma 4.4.7. *Let $\{U_k\}_{k=1}^{\infty}$ be a decreasing sequence of relatively open sets in $\overline{\Omega}^P$ such that $\overline{C_p^P}(U_k) < 2^{-kp}$. Then there exists a decreasing sequence of non-negative functions $\{\psi_j\}_{j=1}^{\infty}$ on Ω such that $||\psi_j||_{N^{1,p}(\Omega)} < 2^{-j}$, $\psi_j \geq k-j$ in $U_k \cap \Omega$ and*

$$\lim_{\Omega \ni y \overset{\overline{\Omega}^P}{\to} x} \psi_j(x) \geq k-j \text{ for all } x \in U_k \cap \partial_P \Omega.$$

We also require a result analogous to Proposition 3.3.8. However, that result was proved purely for bounded Ω. We now prove a similar result for unbounded Ω.

Proposition 4.4.8. *Let $\{f_j\}_{j=1}^{\infty}$ be a p-quasieverywhere decreasing sequence of functions in $D^{1,p}(\Omega)$ such that $f_j \to f$ in $N^{1,p}(\Omega)$. Then Hf_j decreases to Hf in Ω. If u and u_j are solutions to the $\mathcal{K}_{f,f}$ and $\mathcal{K}_{f_j,f_j}$-obstacle problems respectively, then $\{u_j\}_{j=1}^{\infty}$ decreases quasieverywhere in Ω to u.*

Proof. We know each Hf_j to be a minimizer. Thus, $\lim_{j\to\infty} Hf_j$ itself is a minimizer.

We now show that $f - \lim\limits_{j\to\infty} Hf_j \in D_0^{1,p}(\Omega)$. Consider the sequence $\{f_j - Hf_j\}_j$ in $D_0^{1,p}(\Omega)$. Then

$$\begin{aligned}
||g_{f_j - Hf_j}||_{D^{1,p}(\Omega)} &= \left(\int_\Omega g_{f_j - Hf_j}^p \, d\mu \right)^{1/p} \\
&\leq \left(\int_\Omega g_{f_j}^p \, d\mu \right)^{1/p} + \left(\int_\Omega g_{Hf_j}^p \, d\mu \right)^{1/p} \\
&\leq 2 \left(\int_\Omega g_{f_j}^p \, d\mu \right)^{1/p} \quad \text{(since } ||g_{f_j}||_{D^{1,p}(\Omega)} \leq ||g_{Hf_j}||_{D^{1,p}(\Omega)} \text{ by definition.)}
\end{aligned}$$

Since $\{f_j\}$ converges in $N^{1,p}(\Omega)$, we know that $\{g_{f_j}\}$ is a bounded sequence in $L^p(\Omega)$. Therefore, by the above, $\{g_{f_j - Hf_j}\}$ is also bounded in $L^p(\Omega)$. Since X supports a weak $(1,p)$-Poincaré inequality, and the fact that each $f_j - Hf_j$ is zero p-quasieverywhere outside of Ω, for any point $x_0 \in \Omega$

$$\left(\int\limits_{B(x_0,R)} |f_j - Hf_j|^p \, d\mu \right)^{1/p} \leq CR \left(\int\limits_{B(x_0,\lambda R)} g_{f_j - Hf_j}^p \, d\mu \right)^{1/p},$$

and thus $\{f_j - Hf_j\}$ is bounded in $N^{1,p}_{loc}(\Omega)$. Thus, by Lemma 4.3.5, we see that $f - \lim\limits_{j\to\infty} Hf_j \in N^{1,p}_{loc}(\Omega)$. In addition,

$$\int\limits_U g_{f - \lim_{j\to\infty} Hf_j} \, d\mu \leq \liminf_{j\to\infty} \int\limits_U g_{f_j - Hf_j} \, d\mu$$

for any bounded open set $U \subset \Omega$. Thus

$$\int\limits_\Omega g_{f - \lim_{j\to\infty} Hf_j} \, d\mu \leq \liminf_{j\to\infty} \int\limits_\Omega g_{f_j - Hf_j} \, d\mu$$

and $g_{f - \lim_{j\to\infty} Hf_j} \in D^{1,p}_0(\Omega)$. Therefore, we see that $\lim\limits_{j\to\infty} Hf_j$ is a solution to the $\mathcal{K}_{-\infty,f}(\Omega)$-obstacle problem. By the uniqueness of this solution, $\lim\limits_{j\to\infty} Hf_j = Hf$. By our comparison principle above, since f_j is a decreasing sequence, so must Hf_j be.

A repetition of the above argument with u_j and u in place of Hf_j and Hf shows the second part of the proposition. $\square$

We now state our main theorem.

Theorem 4.4.9. *Let $f \colon \overline{\Omega}^P \to \overline{\mathbb{R}}$ be a $\overline{C}_p^P$-quasicontinuous function such that $f|_\Omega$ is in $N^{1,p}(\Omega)$ and the zero extension of $f - Hf$ to $\partial_P\Omega$ is $\overline{C_p^P}$-quasicontinuous. Then f is resolutive and $P_{\overline{\Omega}^P} f = Hf$.*

Proof. First, we assume that $f \geq 0$. We extend Hf to $\overline{\Omega}^P$ by letting $Hf = f$ on $\partial_P\Omega$.

Let $h = f - Hf$. Then, by assumption, $h \in D^{1,p}_0(\Omega)$ is quasicontinuous on Ω with $\overline{C_p^P}$-quasicontinuous extension $h = 0$ to $\partial_P\Omega$. Because f is $\overline{C_p^P}$-quasicontinuous on

$\overline{\Omega}^P$, it now follows that so is Hf.

Pick open sets $\{G_j\}$ in $\overline{\Omega}^P$ with $\overline{C_p^P}(G_j) < 2^{-jp}$ such that $Hf|_{\overline{\Omega}^P \setminus G_j}$ is continuous. Defining $U_k = \displaystyle\bigcup_{j=k+1}^{\infty} G_j$, we see that $\overline{C_p^P}(U_k) < 2^{-kp}$ and $Hf|_{\overline{\Omega}^P \setminus U_k}$ is still continuous.

These sets $\{U_k\}$ fulfill the conditions of Lemma 3.3.7, and so we may take functions $\{\psi_j\}$ as described in that Lemma. We set $f_j = Hf + \psi_j$ (note here that f_j is a function on Ω alone) and let ϕ_j be the lower semicontinuously regularized solution of the $\mathcal{K}_{f_j,f_j}(\Omega)$-obstacle problem.

For each positive integer m we have that

$$f_j \geq \psi_j \geq m \text{ on } U_{m+j} \cap \Omega.$$

Given $\varepsilon > 0$, let $x \in \partial_P \Omega$. If $x \notin U_{m+j}$, by the continuity of $Hf|_{\overline{\Omega}^P \setminus U_{m+j}}$, there is a neighborhood V_x of x in $\overline{\Omega}^P$ such that

$$f_j(y) \geq Hf(y) \geq Hf(x) - \varepsilon = f(x) - \varepsilon \text{ for all } y \in (V_x \cap \Omega) \setminus U_{m+j}.$$

So, if $x \in \partial_P \Omega \setminus U_{m+j}$,

$$f_j \geq \min\{f(x) - \varepsilon, m\} \text{ in } V_x \cap \Omega^P.$$

If, instead, $x \in U_{m+j}$, we take $V_x = U_{m+j}$.

Now, by the previous paragraphs,

$$f_j \geq \min\{f(x) - \varepsilon, m\} \text{ in } V_x \cap \Omega.$$

Since ϕ_j is the solution of the $\mathcal{K}_{f_j,f_j}$-obstacle problem, $\phi_j \geq f_j$ quasieverywhere. Therefore, $\phi_j(y) \geq \min\{f(x) - \varepsilon, m\}$ quasieverywhere in $V_x \cap \Omega$. However, ϕ_j is lower semicontinuously regularized, and hence $\phi_j \geq f_j$ everywhere. Thus, $\phi_j(y) \geq$

$\min\{f(x)-\varepsilon, m\}$ for all $y \in V_x \cap \Omega$. Therefore,

$$\liminf_{\Omega \ni y \overset{\overline{\Omega}^P}{\to} x} \phi_j(y) \geq \min\{f(x)-\varepsilon, m\}.$$

As $\varepsilon \to 0$ and $m \to \infty$, we have that

$$\liminf_{\Omega \ni y \overset{\overline{\Omega}^P}{\to} x} \phi_j(y) \geq f(x) \text{ for all } x \in \partial_P \Omega.$$

Since ϕ_j is p-superharmonic, we have that $\phi_j \in \mathcal{U}_f(\overline{\Omega}^P)$, and so $\phi_j \geq \overline{P}_{\overline{\Omega}^P} f$. Because Hf is the solution to the $\mathcal{K}_{Hf,Hf}(\Omega)$-obstacle problem, by Proposition 4.4.8 we know that ϕ_j decreases quasieverywhere to Hf, that is, $\overline{P}_{\overline{\Omega}^P} f \leq Hf$ q.e. in Ω when $f \geq 0$.

Note that if $f \in D^{1,p}(\Omega)$ has a $\overline{C_p^P}$-quasicontinuous extension to $\overline{\Omega}^P$, then so does $\max\{f, m\}$ for each integer m. Therefore, for $f \in D^{1,p}(\Omega)$, not necessarily non-negative,

$$\overline{P}_{\overline{\Omega}^P} f \leq \lim_{m \to -\infty} \overline{P}_{\overline{\Omega}^P} \max\{f, m\} \leq \lim_{m \to \infty} H \max\{f, m\} = Hf \quad \text{q.e. in } \Omega.$$

Because $\overline{P}_{\overline{\Omega}^P} f$ is p-harmonic in Ω and hence is continuous, we have that both $\overline{P}_{\overline{\Omega}^P} f$ and Hf are continuous. Therefore $\overline{P}_{\overline{\Omega}^P} f \leq Hf$ everywhere in Ω.

Finally, with the aid of our comparison principle proved earlier,

$$\underline{P}_{\overline{\Omega}^P} f = -\overline{P}_{\overline{\Omega}^P}(-f) \geq -H(-f) = Hf \geq \overline{P}_{\overline{\Omega}^P} f \geq \underline{P}_{\overline{\Omega}^P} f.$$

Thus $Hf = \underline{P}_{\overline{\Omega}^P} f = \overline{P}_{\overline{\Omega}^P} f$ and f is resolutive. $\square$

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